Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 20, 2002*
Revised: *December 14, 2002*
Accepted: *December 20, 2002*

Renormalization and asymptotic safety in truncated quantum Einstein gravity

Max Niedermaier
Laboratoire de Mathématiques et Physique Théorique
CNRS/UMR 6083, Université de Tours
Parc de Grandmont, 37200 Tours, France
E-mail: max@phys.univ-tours.fr

Abstract: A perturbative quantum theory of the 2-Killing vector reduction of general relativity is constructed. Although non-renormalizable in the standard sense, we show that to all orders of the loop expansion strict cut-off independence can be achieved in a space of lagrangians differing only by a field dependent conformal factor. In particular the Noether currents and the quantum constraints can be defined as finite composite operators. The form of the field-dependence in the conformal factor changes with the renormalization scale and a closed formula is obtained for the beta functional governing its flow. The flow possesses a unique fixed point at which the trace anomaly is shown to vanish. The approach to the fixed point adheres to Weinberg's "asymptotic safety" scenario, both in the gravitational wave/cosmological sector and in the stationary sector.

Keywords: Renormalization Group, Sigma Models, Models of Quantum Gravity.

Contents

1. Introduction and survey

The hope that quantum Einstein gravity, although perturbatively non-renormalizable [1, 2, 3, 4] is in some sense renormalizable beyond perturbation theory underlies most approaches toward constructing such a theory. Explicitly this hope is expressed in S. Weinberg's "asymptotic safety" scenario [1]. Implicitly however it also underlies the modern background independent approaches, such as the dynamical triangulations approach [5] or the canonical quantization program of A. Ashtekar and its ramifications. In these approaches the renormalizability problem seems to reappear in specific, not foreseeable features like the tendency of random geometries to degenerate [5, 6], the singular support of diffeomorphism invariant measures [7], or the scarcity of semi-classical states [8]. The hope that these problems can eventually be overcome in one or more of these approaches has recently been revived by the results of M. Reuter and O. Lauscher [9, 10, 11], reporting non-trivial though not yet compelling evidence for the viability of the original asymptotic safety scenario. See also [12, 13, 14]. The goal of the present work is to gain more detailed insight into the renormalizability issue in a more manageable, truncated version of the theory, corresponding to all spacetimes with two Killing vectors.

The 2-Killing vector reduction has already been widely used as a laboratory for studying quantum aspects of general relativity, see e.g. [30, 32, 35, 29, 28]. In particular for selfinteracting cylindrical gravitational waves an in principle exact 'bootstrap' quantization was proposed in [28]. In that framework the issue of (non-)renormalizability is by-passed. Since it is a reduced phase space quantization based on a bootstrap principle, however, some important issues cannot be addressed. In particular one would like to understand the link to the perturbative divergencies and their bookkeeping, the status of the quantum constraints and their algebra, and the projection onto the physical state space. In other words one would like to have a more-or-less conventional quantum field theoretical framework in which a Dirac quantization program could be implemented. This is what we set out to initiate here.

The reduced phase turns out to be equivalent to that of two-dimensional (2D) gravity non-minimally coupled via a 'radion' field ρ to a 2D matter system. The radion field is related to the determinant of the internal metric, the matter system is a noncompact $\mathrm{O}(1,2)$ nonlinear sigma-model. This means the matter fields are maps $n = (n^0, n^1, n^2)$ from a 2D manifold into the hyperboloid $H_+ = \{n = (n^0, n^1, n^2) \in \mathbb{R}^{1,2} \,|\, n\cdot n = (n^0)^2 - (n^1)^2 - (n^2)^2 = 1,\, n^0 > 0\}$. A sign $\epsilon = \pm 1$ will distinguish the two main situations, where either both Killing vectors are spacelike ($\epsilon = +1$) or one is spacelike and the other timelike ($\epsilon = -1$). Accordingly the 2D metric $\gamma_{\mu\nu}$ will have signature $1 - \epsilon$; without (much) loss of generality we shall always assume it to be conformally flat $\gamma_{\mu\nu} \sim e^{\sigma}\eta_{\mu\nu}$, and describe the dynamics in terms of σ. The reduced phase space is then characterized by the equations of motion and the symplectic structure following from the flat space action $S = \int d^2x L$, with

$$L(n, \rho, \sigma) = -\frac{1}{2\lambda}[\rho\partial^\mu n \cdot \partial_\mu n + \partial^\mu \rho \partial_\mu (2\sigma + \ln \rho)]\,, \qquad n \cdot n = 1\,, \tag{1.1}$$

where λ is Newton's constant per unit volume of the internal space. As explained in appendix A both sectors $\epsilon = \pm 1$ can be described by the action (1.1) but the 4D spacetime interpretation of the fields is very different. Further, to account for the original 2D diffeomorphism invariance the weak vanishing of the hamiltonian and the diffeomorphism constraints $\mathcal{H}_0 \approx 0$, $\mathcal{H}_1 \approx 0$, has to be imposed. The latter are given by $\mathcal{H}_0 = T_{00}$ and $\mathcal{H}_1 = T_{01}$, if $T_{\mu\nu}$ denotes the classical energy momentum tensor derived from (1.1). The trace T^μ_μ vanishes on-shell.

JHEP12(2002)066

Suppose we now want to embark on a Dirac quantization of the system. That is the vector n as well as ρ, σ are promoted to independent quantum fields whose dynamics is governed by the action (1.1). The indefiniteness in its ρ, σ part reflects the stability problem related to (or replacing) the "conformal factor" problem of 4D quantum Einstein gravity [16]. The associated negative norm states hopefully will decouple from the physical state space, defined schematically as the 'kernel' of the constraints. The constraints ought to be defined as composite operators through a generalized action principle. Clearly the key issue to be addressed then is that of the renormalizability of an action functional that is motivated by (1.1) and its symmetries and extended by suitable sources needed to define composite operators. In contrast to nonlinear sigma-models without coupling to gravity we find that the quantum field theory based on (1.1) is *not* ultraviolet renormalizable in the standard sense.

Renormalizability in the standard quantum field theoretical sense typically presupposes that the bare and the renormalized (source extended) action have the same functional form, only the arguments of that functional (fields, sources, and coupling constants) get renormalized. Though the bare action is motivated by the classical one it can be very different from it. In any case the form of the (bare=renormalized) action functional is meant to be known before one initiates the renormalization. The 2-Killing vector reduction has been known for some time to be 1-loop renormalizable [36]. Here we confirm this result, but we also find that the system is not renormalizable in the above sense beyond 1-loop.[1] The reason is that the dependence on the dimensionless 'radion' field ρ is not constrained by any Noether symmetry. Thus it can — and does — enter the counter terms in a different way as in the bare action, no matter how the latter is chosen.

The solution we propose is to renormalize the theory in a space of lagrangians differing by an overall conformal factor that is a function of ρ [17]. More precisely we show that to *all orders* in the loop expansion nonlinear field renormalizations exist such that for any prescribed bare $h_B(\,\cdot\,)$ there exists a renormalized $h(\,\cdot\,)$ such that

$$\frac{h_B(\rho_B)}{\rho_B} L(n_B, \rho_B, \sigma_B) = \frac{h(\rho)}{\rho} L(n, \rho, \sigma)\,, \qquad \text{but} \qquad h_B(\,\cdot\,) \neq h(\,\cdot\,)\,. \tag{1.2}$$

A subscript '$_B$' denotes the bare fields while the plain symbols refer to the renormalized ones. The fact that $h_B(\,\cdot\,)$ and $h(\,\cdot\,)$ differ marks the deviation from conventional renormalizability; it holds with one notable exception described in appendix C. In order to be able to perform explicit computations we show this result in a specific computational scheme: dimensional regularization, minimal subtraction and the covariant background field expansion. We expect however that the main features remain valid in a regularization independent analysis along the lines of [46]; cf. section 3. We made no attempt to address the infrared problem here because, guided by the analogy to the abelian sector [34], we expect it to disappear upon projection onto the physical state space.

In view of (1.2) the function $h(\,\cdot\,)$ plays the role of a generalized (renormalized) coupling, which is "essential" in the sense of [1]. As such it is subject to a flow equation

[1]The reduced system thus accurately portrays the features of full quantum Einstein gravity [2, 3, 4].

JHEP12(2002)066

$\mu \frac{d}{d\mu}\overline{h} = \beta_h(\overline{h}/\lambda)$, where μ is the renormalization scale and $\mu \to \overline{h}(\,\cdot\,,\mu)$ is the 'running' coupling function. Remarkably $\beta_h(h)$ can be obtained in closed form and is given by

$$\beta_h(h/\lambda) = -\rho\partial_\rho\left[\frac{h(\rho)}{\lambda}\int_\rho^\infty \frac{du}{u}\frac{h(u)}{\lambda}\beta_\lambda\Big(\frac{\lambda}{h(u)}\Big)\right]. \tag{1.3}$$

Here $\beta_\lambda(\lambda)$ is the conventional (numerical) beta function of the O(1, 2) nonlinear sigma-model without coupling to gravity, computed in the minimal subtraction scheme. $\beta_h(h)$ can thus be regarded as a "gravitationally dressed" version of $\beta_\lambda(\lambda)$, akin to the phenomenon found in [73]. The $\overline{h}$-flow turns out to have a unique *UV stable fixed-point* $h^{\rm beta}$, satisfying $\beta_h(h^{\rm beta}) = 0$. This establishes Weinberg's asymptotic safety scenario in the truncation considered.

We proceed by showing that the energy momentum tensor can be defined as a finite composite operator $[\![T_{\mu\nu}]\!]$ by adding a judiciously chosen improvement term $\Delta T_{\mu\nu} = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(f(\rho) + f_0\,\sigma)$. After renormalization the latter is characterized by a function $f(\,\cdot\,)$ and a constant f_0. Scale changes then trigger a non-autonomous inhomogeneous flow equation for a running $\overline{f}(\,\cdot\,,\mu)$. Clearly a necessary condition for the quantum constraints $[\![T_{00}]\!]$ and $[\![T_{01}]\!]$ to have a sufficiently large 'kernel' on the state space generated by $\rho, \partial_\mu\sigma, n^i\partial_\mu n^j$ is that the trace anomaly $[\![T^\mu{}_\mu]\!]$ vanishes modulo contributions proportional to the equations of motion operator. One might expect that when $\overline{h}$ becomes stationary also $\overline{f}$ becomes stationary and defines the proper improvement potential at the fixed point. This is indeed the case, moreover

$$\mu\frac{d}{d\mu}\overline{h} = 0 = \mu\frac{d}{d\mu}\overline{f} \quad\Longleftrightarrow\quad [\![T^\mu_\mu]\!] = 0\,, \tag{1.4}$$

to all loop orders. That is, the trace anomaly of the improved energy momentum tensor vanishes precisely at the fixed point of the functional flow.

The article is organized as follows: in the next section we motivate the rescaled lagrangians in (1.1) via Weyl transformations in the classical theory and discuss heuristic aspects of the quantization procedure in both $\epsilon = \pm 1$ sectors. Section 3 provides the counter terms through a reformulation as a riemannian sigma-model. The main renormalization architecture is laid out in section 4 at a fixed renormalization scale μ. Variation of μ induces flow equations whose fixed point structure is investigated subsequently. Section 6 establishes the link to the vanishing of the trace anomaly alluded to. We conclude with some directions for future research.

2. Putting conformal factors into action

For the 4D Einstein-Hilbert lagrangian we adopt the normalization

$$L(G) = -\frac{1}{\lambda^{(4)}}\sqrt{-\det G}\,R^{(4)}(G)\,, \tag{2.1}$$

with the conventions detailed in appendix A. Inserting the generic form (A.1), (A.3) of a metric with two Killing vectors it turns into the lagrangian of the truncated theory we

JHEP12(2002)066

wish to explore:

$$L_\epsilon(n,\rho,\sigma) = -\frac{1}{2\lambda}\rho\sqrt{\gamma}\gamma^{\mu\nu}[\partial_\mu n\cdot\partial_\nu n+\epsilon\rho^{-2}\partial_\mu\rho\partial_\nu\rho]-\frac{\epsilon}{\lambda}\rho\sqrt{\gamma}R^{(2)}(\gamma)+\frac{2\epsilon}{\lambda}\sqrt{\gamma}\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\rho\,. \tag{2.2}$$

Here all fields are functions of the non-Killing coordinates (x^0, x^1) only. The coupling constant $\lambda > 0$ is Newton's constant per unit volume of the internal space. Further ∇_μ is the covariant derivative with respect to $\gamma_{\mu\nu}$, which we assume to be diffeomorphic to $\eta_{\mu\nu}e^\sigma$ throughout. The 2D curvature is normalized such that $R^{(2)}(e^\sigma\eta) = -e^{-\sigma}\partial^2\sigma$. The last term in (2.2) is a total derivative which of course could be discarded in the action. We keep it here because of its nontrivial interplay with conformal transformations of the 4D metric and the 2D metric.

The induced 'matter' part $-\epsilon\partial_\mu n\cdot\partial_\nu n = (\partial_\mu\Delta\partial_\nu\Delta + \epsilon\partial_\mu B\partial_\nu B)/\Delta^2$ is the pull-back of the canonical metric on the hyperboloid H_ϵ in (A.4). For $\epsilon = -1$ the induced 'matter' fails to have good positivity properties. However there exists a dual (classically equivalent) lagrangian L^D_ϵ such that

$$\begin{aligned} L^D_\epsilon(\eta(\phi_D)) &= \epsilon L_+(\phi_D)\,, \\ \eta(\Delta, B_D, \rho, \sigma) &= \Big(\frac{\rho}{\Delta}, B_D, \rho, \sigma + \frac{1}{2}\ln\rho - \ln\Delta\Big)\,, \qquad \eta^2 = \mathrm{id}\,. \end{aligned} \tag{2.3}$$

Here $\partial_\mu B_D = -\epsilon\rho\Delta^{-2}\epsilon_{\mu\nu}\partial^\nu B$, on-shell, and $\phi_D = (\Delta, B_D, \rho, \sigma)$ denotes the collection of fields. Thus also for the stationary sector, $\epsilon = -1$, can one use an action with a definite 'matter' part, just that one of fields is (on-shell) nonlocally related to the one appearing in the line element. The transition from L_ϵ to L^D_ϵ is an instance of abelian T-duality in the sense of Buscher [65]; the involution η is closely related to the Kramer-Neugebauer involution [20]; cf. appendix A and section 2.2. Apart from the heuristic discussion in section 2.2 we shall exclusively use the lagrangian L_ϵ, keeping in mind that L_+ can be used to describe both the gravitational wave/cosmological sector ($\epsilon = +1$) and the stationary axisymmetric sector ($\epsilon = -1$). The case L_- will be carried along, mainly in order to highlight the differences.

In summary the reduced lagrangian is that of a 2D gravity theory non-minimally coupled via ρ to a non-compact $\mathrm{O}(1,2)$ nonlinear sigma model. It is important that the step from (2.1) to (2.2) is a "symplectic reduction", i.e. both the reduced Einstein equations and the symplectic structure induced from the 4D theory coincide with the ones derived from the action $S_\epsilon = \int d^2x\, L_\epsilon$.

2.1 4D and 2D Weyl transformations — generalized Ernst systems

The rescaled lagrangians instrumental for the renormalization process via (1.2) have a natural classical counterpart: if $G_{MN}(x)$ denotes a generic 4D metric with two Killing vectors (in adapted coordinates) the relevant structure are Weyl transformations of the form

$$G_{MN}(x) \;\longrightarrow\; \exp\omega(\rho(x))\, G_{MN}(x)\,, \tag{2.4}$$

where x are the non-Killing coordinates and $\rho(x)$ is the 'radion' field related to the determinant of the internal metric. Weyl transformations where the spacetime dependence of the conformal factor enters through a function of a scalar field are frequently used in scalar-tensor theories of gravity and in the dimensional reduction of gravity theories; see e.g. [26] for a review.

Clearly a x-dependent rescaling of the 4D metric, $G_{MN}(x) \to e^{\omega(x)}\, G_{MN}(x)$, maps one metric with two Killing vectors onto another one. In the parameterization (A.1) this amounts to the simultaneous replacement $\gamma_{\mu\nu} \to e^{\omega}\gamma_{\mu\nu}$, $\rho \to e^{\omega}\rho$, while $\gamma_{\mu\nu} \to e^{\omega}\gamma_{\mu\nu}$, $\rho \to \rho$ are 2D Weyl transformations. From (2.2) one computes

$$\begin{aligned} &\text{2D}: \qquad L_\epsilon(n,\rho,\omega+\sigma) = L_\epsilon(n,\rho,\sigma) + \frac{\epsilon}{\lambda}\rho\sqrt{\gamma}\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\omega\,,\\ &\text{4D}: \qquad L_\epsilon(n,e^{\omega}\rho,\omega+\sigma) = e^{\omega}L_\epsilon(n,\rho,\sigma) + \frac{6\epsilon}{\lambda}\sqrt{\gamma}\gamma^{\mu\nu}e^{\omega/2}\nabla_\mu(\rho\nabla_\nu e^{\omega/2})\,. \end{aligned} \tag{2.5}$$

The first equation implies in particular that the action is invariant under a restricted class of ('conformal') 2D Weyl transformations $\gamma_{\mu\nu} \to e^{\omega}\gamma_{\mu\nu}$ with $\nabla^2\omega = 0$. This also explains why the trace $T^{\mu}{}_{\mu}$ vanishes only on-shell. The 2D Weyl transformations can also be used to generate non-vacuum solutions of Einstein's equations with two Killing vectors from vacuum solutions [25]. In the present context 4D Weyl transformation of the form (2.4) turn out to be important.

Consider the following generalization of the lagrangian (2.2) or (A.8)

$$\begin{aligned} L_{hab} &= \frac{1}{2\lambda}h(\rho)\sqrt{\gamma}\gamma^{\mu\nu}[-\partial_\mu n\cdot\partial_\nu n + \rho^{-2}a(\rho)\partial_\mu\rho\partial_\nu\rho] + \frac{1}{2\lambda}f(\rho)\sqrt{\gamma}R^{(2)}(\gamma)\\ &\simeq \frac{1}{2\lambda}h(\rho)[-\partial^\mu n\cdot\partial_\mu n + a(\rho)\rho^{-2}\partial^\mu\rho\partial_\mu\rho + 2b(\rho)\rho^{-1}\partial^\mu\rho\partial_\mu\sigma]\,. \end{aligned} \tag{2.6}$$

Here we introduced arbitrary functions $h(\rho), a(\rho)$ and $b(\rho)$ such that

$$f(\rho) = 2\int^{\rho}\frac{du}{u}h(u)b(u)\,, \tag{2.7}$$

and omitted total derivative terms. The constraints read

$$\lambda T^{hab}_{\pm\pm} = h(\rho)[-\partial_\pm n\cdot\partial_\pm n + \rho^{-2}a(\rho)(\partial_\pm\rho)^2] + \partial_\pm\sigma\partial_\pm f - \partial_\pm^2 f\,. \tag{2.8}$$

while $\lambda T^{hab}_{+-} = \partial_+\partial_- f$ vanishes on account of the equations of motion $\partial^\mu\partial_\mu f = 0$.

The classical systems (2.6)–(2.8) are related to the original one in two ways: first for $a(\rho), b(\rho)$ given by specific expressions in terms of $h(\rho)$ the former is the 4D Weyl transform of the latter, with $e^{\omega} = h(\rho)/\rho$ [17]. Indeed, in conformal gauge this amounts to the substitution

$$\text{4D}: \qquad \rho \longrightarrow h(\rho)\,, \quad \sigma \longrightarrow \sigma + \ln[h(\rho)/\rho]\,. \tag{2.9}$$

Discarding total derivatives one finds

$$L_\epsilon(n,h,\sigma+\ln h/\rho) \simeq L_{hab}$$

with

$$a(\rho) = -\epsilon[3(\rho\partial_\rho\ln h)^2 - 2\rho\partial_\rho\ln h]\,, \qquad b(\rho) = -\epsilon\rho\partial_\rho\ln h\,. \tag{2.10}$$

Specifically for $h(\rho) = \rho^p$ the functions $a(\rho)$, $b(\rho)$ again reduce to (shifted) constants: $a = -\epsilon(3p^2 - 2p)$, $b = -\epsilon p$.

JHEP12(2002)066

Secondly, for generic $a(\rho)$ and $b(\rho)$ unrelated to $h(\rho)$ one can use field redefinitions to simplify the lagrangian (2.6) such that it differs from the one in (1.1) only by an overall factor $h(\rho)/\rho$. The explicit transformation is a by-product of the considerations in section 3.1; we thus postpone its description to eq. (3.10). The class of lagrangians L_{hab} in (2.6) with constant a, b turns out to be the appropriate setting for the renormalization. Before turning to this, however, some heuristics is called for.

2.2 Quantization — heuristics

Morally speaking one would like to make sense out of the functional integral

$$(a) \quad \int_{2-\mathrm{Killing}} \mathcal{D}G \, e^{iS[G]}$$

or

$$(b) \quad \int \mathcal{D}n\mathcal{D}\rho\mathcal{D}\sigma \, e^{id^2y \int d^2x \, L_\epsilon(n,\rho,\sigma)} \quad \text{with} \quad \mathcal{H}_0 \simeq \mathcal{H}_1 \simeq 0 \, . \tag{2.11}$$

In the first version the intended functional integral ranges over all 4D lorentzian metrics with two Killing vectors of fixed signature and fixed unit volume d^2y of the internal space. $S[G]$ is the Einstein-Hilbert action. Of course (a) cannot be expected to be "the same" as "first quantizing and then truncating." Also, but not only, because the latter lacks a precise meaning so far. However in the context of the asymptotic safety scenario the truncation should preserve the existence and the qualitative features of a fixed point: if the full theory is assumed to have an UV stable fixed point every truncation that preserves the presumed 'ferromagnetic' or 'anti-screening' [13, 11] nature of the self-coupling should likewise display such a fixed point (but not vice versa). Simply 'freezing' the fluctuations transversal to the Killing orbits should meet this condition. The search for an UV stable fixed point in (a) and (b) thus provides an important self-consistency test for the asymptotic safety scenario.

In version (b) of the functional integral part of the 4D reparameterization invariance has been fixed by choosing the parameterization (A.1) of G_{MN} adapted to the Killing vectors and the residual 2D diffeomorphism invariance has been partially fixed at the expense of the constraints. In this version one aims at a Dirac quantization of the system; it is the one which we investigate in the bulk of the paper.

The reason for spelling out both intentions (2.11) is that reasonable implementations of (a) always lead to versions of (b) where the 'matter' part of the reduced lagrangian corresponds to a sigma-model with a riemannian (rather than pseudo-riemannian) target space, i.e. the hyperboloid H_+ in (A.4). The resulting functional integral in (b) can then be interpreted either as modeling a subsector of 4D lorentzian quantum gravity or as a subsector of 4D euclidean quantum gravity.

For $\epsilon = +1$ this conclusion is obvious. Both the original action (A.8) and the dual action (A.10) have 2D 'matter' sectors based on the hyperboloid H_+. Both follow directly from a 4D line element, namely (A.6) and (A.12), respectively. The base space of the 2D reduced actions has Minkowski signature and the functional integral (b) has its usual quantum field theoretical meaning. Since the lagrangians are Poincaré invariant and we confine ourselves to a perturbative study a Wick rotation is legitimate, technically convenient, but not essential. Looking back at (A.6) or (A.12) one sees that a Wick rotation of x^0 gives the 4D line elements a riemannian signature, $(-,-,-,-)$. The reduced lagrangians can in both versions be written in the form

$$L_E(n,\rho,\sigma) = \frac{1}{2\lambda}\left[\rho\frac{(\partial_\mu\Delta)^2 + (\partial_\mu B)^2}{\Delta^2} - \partial_\mu\rho\partial_\mu(2\sigma + \ln\rho)\right]. \tag{2.12}$$

This is readily checked to coincide with the result of the 2-Killing reduction of the Einstein-Hilbert action for riemannian metrics of signature $(+,+,+,+)$. As expected, the $\mathrm{O}(1,2)$ part is manifestly positive, while the indefiniteness of the $\partial_\mu\rho\partial_\mu\sigma$ part (manifest by diagonalization) is reflects the stability problem related to (or replacing) the "conformal factor" problem [16]. The exponential now looks more appealing, $\exp(-d^2y\int d^2x\, L_E)$, but in a perturbative context it is equivalent to the original $\exp(i\, d^2y\int d^2x\, L_+)$ of (b).

In contrast, when starting from the stationary axisymmetric line element, i.e. eq. (A.6) or (A.12) with $\epsilon = -1$, the 2D matter sector in the reduced actions is indefinite. However the 2D base space is already euclidean, so that $\exp(i\, d^2y\int d^2x\, L_-)$ has no immediate quantum field theoretical interpretation. Such an interpretation can be restored in two ways. One proceeds by analytic continuation and the other by dualization. Both are merely heuristic but both give rise to a 2D 'matter' sigma-model based on the hyperboloid H_+. In the first argument one notes that upon analytic continuation in the Killing time y^2 the functional integral regains a quantum field theoretical meaning. However the line element (A.6) then is complex, unless one performs an additional replacement $B \to iB$ (or restricts attention to the abelian $B = 0$ sector). Alas, doing both replacements $y^2 \to iy^2$ and $B \to iB$ gives back (2.12). This argument is somewhat unsatisfactory because replacing a real variable in a functional integral by a purely imaginary one can hardly be expected to be legitimate. A safe conclusion would be that no conclusion can be drawn based on the actions obtained by direct reduction in the axisymmetric sector.

The second argument is better and proceeds by dualization. As outlined in appendix A, applying an abelian T-duality transformation to the lagrangian L_- in (2.2) leads to the classically equivalent lagrangian (A.10) where the spins live on H_+. Adapting the familiar functional integral argument [65] to the case at hand this comes about as follows. Consider

$$\begin{aligned}
&\int \mathcal{D}\Delta\mathcal{D}V\mathcal{D}B_D \exp\left\{i\, d^2y\int d^2x\, L^{\mathrm{gauge}}\right\}, \\
&L^{\mathrm{gauge}} = \frac{1}{2\lambda}\left[\rho\Delta^{-2}(-(\partial_\mu\Delta)^2 + V_\mu^2) + 2B_D\,\epsilon^{\mu\nu}\partial_\mu V_\nu - \rho^{-1}(\partial_\mu\rho)^2 - 2\partial_\mu\rho\partial_\mu\sigma\right].
\end{aligned} \tag{2.13}$$

The lagrangian can be viewed as a gauged version of L_- by decomposing V_μ as $V_\mu = A_\mu + \partial_\mu B$, with $\delta B = \lambda(x)$, $\delta A_\mu = -\partial_\mu\lambda(x)$. The overall '$i$' stems from the original (heuristic) functional integral (a) over all 4D lorentzian metrics with one timelike and one spacelike Killing vector. Integrating over the Lagrange multiplier field B_D implements the $\epsilon^{\mu\nu}\partial_\mu V_\nu = 0$ constraint, i.e. $V_\mu = \partial_\mu B$ locally, and hence (in the gauged interpretation after gauge fixing) gives back the original reduced functional integral. On the other hand one can perform the gaussian integral over V_μ which (after dropping a divergent measure factor) yields

$$\int \mathcal{D}\Delta\mathcal{D}B_{\scriptscriptstyle D} \exp\{i\, d^2y \int d^2x\, L_-^{\scriptscriptstyle D}\} = \int \mathcal{D}\Delta_{\scriptscriptstyle D}\mathcal{D}B_{\scriptscriptstyle D} \exp\left\{-i\, d^2y \int d^2x\, L_+\right\}, \qquad (2.14)$$

using (A.11) in the second step. As noted in (A.13) the dual action is no longer directly related to a 4D line element and is therefore unaffected by the analytic continuation in the Killing time y^2 that restores a standard quantum field theoretical interpretation of (b). The discrepancy between the result (A.10) for $\epsilon = -1$ and the standard duality formula (where no sign flip occurs in the relevant component of the target space metric $g^{\scriptscriptstyle D}_{BB} = +1/g_{BB}$) arises because in [65, 66] the overall 'i' in (2.13) is replaced with an overall 'minus' (as required by euclidean quantum field theory) and $B_{\scriptscriptstyle D}$ with $iB_{\scriptscriptstyle D}$. We conclude that after dualization the formal functional integral relevant for the stationary axisymmetric subsector is likewise one governed by a sigma-model lagrangian whose 'matter' part corresponds to the riemannian symmetric space H_+ rather than the pseudo-riemannian H_- entering via the direct reduction. Due to the formal nature of the argument (and the fact that both systems are not renormalizable in the standard sense) one cannot expect that the orginal and the dual theory are equivalent. One is thus free to take the dual version (2.14) as the starting point for the construction of a perturbative quantum theory.

Our main goal will therefore be to construct a finite perturbative measure based on the exponentiated lagrangian $\exp(i\int d^2xL_+)$ or $\exp(-\int d^2xL_E)$. For $\epsilon = +1$ this lagrangian can be taken to be the one (A.8) obtained by direct reduction. For $\epsilon = -1$ it should be interpreted as the dualized lagrangian in (A.10), (A.11); for simplicity we drop the subscripts '$\scriptscriptstyle D$' on the fields throughout. However one should keep in mind that the fields in the L_+ lagrangian when viewed as describing the stationary axisymmetric sector have a very *different* 4D spacetime meaning than in the $\epsilon = +1$ sector.

In order to highlight the differences to the H_- case we carry it along in the version $\exp(-\int d^2xL_-)$, where the quantum field theoretical meaning is restored while giving up a link to 4D quantum gravity. Of course the H_- system can still be interpreted as 2D euclidean (conformally flat) quantum gravity non-minimally coupled to a hyperbolic sigma-model.

For the Ernst-like systems so far only quantizations of the reduced phase space have been investigated, see e.g. [35, 29, 28]. As stressed in the introduction a number of important issues can however only be addressed in a Dirac quantization program. A Dirac approach also allows one to decompose the full problem into simpler subproblems: (i) Construction of a finite perturbative measure for the basic lagrangian. (ii) Extension by sources according to the composite operators aimed at — which should include at least the constraints and the Noether currents. (iii) Projection onto the physical state space. In this article we focus on steps (i) and (ii). Concerning (i) we find that although the quantum systems based on (b) in (2.11) are *not* renormalizable in the standard quantum field theoretical sense, they can be rendered renormalizable in the 'conformal' sense outlined in the introduction. This also allows one to define composite operators in a systematic way. In particular, quantum versions of the constraints can be defined, although the actual construction of physical states and proving the absence of negative norm states will still be difficult. Throughout we address only the ultraviolet aspects because it seems that infrared problems will disappear upon projection onto the physical state space. This picture is suggested by the situation in the abelian systems, where (hopefully) Dirac quantization is equivalent to a reduced phase space quantization, and the latter is free of infrared problems [34]. In other words we hope that an infrared cutoff in the nonabelian systems can be removed after projection onto the physical state space. In the following we thus concentrate on steps (i) and (ii), i.e. on perturbatively defining an UV finite quantum theory based on the (source extended) action (1.1) or its generalization L_{hab} in (2.6).

3. Formulation as a riemannian sigma-model

For the reasons explained above we aim at a Dirac quantization of the system, promoting also ρ and σ to independent quantum fields. As always the raw material for a perturbative quantization are the counter terms. It is convenient to interpret the generalized Ernst systems (2.6) as riemannian sigma-models in the sense of Friedan [52]. Taking advantage of the vast literature on these systems one gets the counter terms almost for free. To preclude a misconception, however, let us stress already here that the 'renormalization architecture' built from these counter terms will be very different from the one used for riemannian sigma-models.

3.1 Conformal geometry of the target space

The Ernst system can be treated as a riemannian sigma-model by promoting ρ and σ to extra coordinates on a 4-dim. auxiliary target space. To this end set

$$\phi^1 = \Delta\,, \quad \phi^2 = B\,, \quad \phi^3 = \rho\,, \quad \phi^4 = \sigma\,,$$
$$\widehat{g}_{ij}(\phi) = \begin{pmatrix} \epsilon\rho/\Delta^2 & 0 & 0 & 0 \\ 0 & \rho/\Delta^2 & 0 & 0 \\ 0 & 0 & a/\rho & b \\ 0 & 0 & b & 0 \end{pmatrix}. \tag{3.1}$$

For $a = b = -\epsilon$ then $\frac{1}{2\lambda}\widehat{g}_{ij}(\phi)\,\partial\phi^i\partial\phi^j$ reproduces the lagrangian of (A.8). In the following we keep $a \in \mathbb{R}$ and $b \neq 0$ as parameters, first as a check on the convention-independence, and second because they might turn into coupling constants.

For the rest of this section we now study the geometry of this 4D target space. First note that $\widehat{g}_{ij}(\phi)$ has signature $(\epsilon, +, +, -)$. Further $\widehat{g}_{ij}(\phi)d\phi^i d\phi^j$ has the structure of a "warped product" of the hyperboloid H_ϵ with $\mathbb{R}^{1,1}$. Indeed

$$\widehat{g}_{ij}(\phi)d\phi^i d\phi^j = e^{u^+}[\epsilon ds^2_{H_\epsilon} + 2b du^+ du^-]\,, \qquad u^+ = \ln\rho\,, \quad u^- = \sigma + \frac{a}{2b}\ln\rho\,. \tag{3.2}$$

The scalar curvature is $R(\widehat{g}) = -2\epsilon/\rho$, so that ρ plays the role of a 'radion' field parameterizing the curvature radius of the target manifold.[2]

[2]Note that although H_ϵ has constant curvature -2 for both signatures $\epsilon = \pm 1$ of the Killing vectors, the curvature of the 4D target space depends on ϵ.

Next we determine the Killing symmetries of the metric (3.1). One readily finds that (3.1) admits the following Killing vectors: $\mathbf{t}_- := \partial_\sigma$ and $\mathbf{e}$, $\mathbf{h}$, $\mathbf{f}$, where

$$\begin{aligned} &\mathbf{e} = \partial_B\,, \qquad \mathbf{h} = 2(B\partial_B + \Delta\partial_\Delta)\,, \qquad \mathbf{f} = (-B^2 + \epsilon\Delta^2)\partial_B - 2B\Delta\partial_\Delta\,, \\ &[\mathbf{h},\mathbf{e}] = -2\mathbf{e}\,, \quad [\mathbf{h},\mathbf{f}] = 2\mathbf{f}\,, \qquad [\mathbf{f},\mathbf{e}] = \mathbf{h}\,, \end{aligned} \tag{3.3}$$

generate the isometries of the hyperboloid H_ϵ. In addition to these proper Killing vectors (3.1) admits two conformal Killing vectors

$$\begin{aligned} \mathbf{t}_+ &= \rho\partial_\rho - \frac{a}{2b}\partial_\sigma\,, \\ \mathbf{d} &= -\rho\ln\rho\,\partial_\rho + \Big(\sigma + \frac{a}{b}\ln\rho\Big)\partial_\sigma = -u^+\mathbf{t}_+ + u^-\mathbf{t}_-\,. \end{aligned} \tag{3.4}$$

Together with $\mathbf{t}_- = \partial_\sigma$ they generate the algebra of isometries of $\mathbb{R}^{1,1}$, i.e. $[\mathbf{t}_+,\mathbf{t}_-] = 0$ and $[\mathbf{d},\mathbf{t}_\pm] = \pm\mathbf{t}_\pm$. (Presumably there exists a relation to the conformal symmetries in [24].) Further $\mathbf{t}_\pm$ are null vectors, $\widehat{g}_{ij}(\phi)\mathbf{t}_\pm^i\mathbf{t}_\pm^j = 0$. For later use we also note the finite transformations and the scaling properties of the line element $ds^2 = \widehat{g}_{ij}(\phi)d\phi^i d\phi^j$:

$$e^{-\ln\Lambda\,\mathbf{t}_+} : (\rho,\sigma) \longrightarrow \Big(\Lambda^{-1}\rho, \sigma + \frac{a}{2b}\ln\Lambda\Big)\,, \qquad ds^2 \longrightarrow \Lambda^{-1}\,ds^2\,, \tag{3.5a}$$

$$e^{-\ln\Lambda\,\mathbf{d}} : (u^+,u^-) \longrightarrow (\Lambda u^+, \Lambda^{-1}u^-)\,, \qquad ds^2 \longrightarrow \rho^{\Lambda-1}\,ds^2\,, \tag{3.5b}$$

with $u^\pm$ as in (3.2).

Conversely one can now ask what is the most general form of a target space metric compatible with these symmetries. This will dictate in what subspace of 4D riemannian metrics the renormalization flow can move. Fixing a coordinate system adapted to the Killing vectors proper, one finds that the generic form of the metric admitting in addition the above conformal Killing vectors is

$$g_{ij}(\phi) = h(\rho)\begin{pmatrix} \epsilon/\Delta^2 & 0 & 0 & 0 \\ 0 & 1/\Delta^2 & 0 & 0 \\ 0 & 0 & a(\rho)/\rho^2 & b(\rho)/\rho \\ 0 & 0 & b(\rho)/\rho & 0 \end{pmatrix}, \tag{3.6}$$

for some functions $h(\rho)$, $a(\rho)$, $b(\rho)$. The corresponding lagrangian is the one anticipated in eq. (2.6). The vanishing of g_{44} in (3.6) ensures that the curvature of the lower 2×2 block vanishes; correspondingly the isometries of $\mathbb{R}^{1,1}$ (acting like conformal Killing vectors on the full metric) are still present. Indeed, the two conformal Killing vectors now read

$$\begin{aligned} \mathbf{t}_+ &= \frac{b}{b(\rho)}\rho\partial_\rho - \frac{ba(\rho)}{2b(\rho)^2}\,\partial_\sigma\,, \\ \mathbf{d} &= -\left\{\frac{\rho}{b(\rho)}\int^\rho \frac{du}{u}b(u)\right\}\partial_\rho + \left\{\sigma + \int^\rho \frac{du}{u}\frac{a(u)}{2b(u)} + \frac{a(\rho)}{2b(\rho)^2}\int^\rho \frac{du}{u}b(u)\right\}\partial_\sigma\,. \end{aligned} \tag{3.7}$$

The scale factors in the conformal Killing equations are

$$\begin{aligned} \mathcal{L}_{\mathbf{t}_+}g_{ij} &= \frac{b\rho\partial_\rho\ln h}{b(\rho)}g_{ij}\,, \\ \mathcal{L}_{\mathbf{d}}g_{ij} &= -\frac{\rho\partial_\rho\ln h}{b(\rho)}\int^\rho \frac{du}{u}b(u)g_{ij}\,. \end{aligned} \tag{3.8}$$

One can also check that $\mathbf{t}_+$, $\mathbf{d}$, and $\mathbf{t}_- = \partial_\sigma$ continue to generate the isometries of $\mathbb{R}^{1,1}$.

Of course in (3.6) one is still free to perform coordinate transformations in the non-Killing coordinate ρ. If we insist that the $\mathbf{t}_-$ Killing vector continues to act like the $j=4$ coordinate derivative the allowed residual transformations are

$$\rho \longrightarrow \widetilde{\phi}^3(\rho) = \tilde{\rho}\,, \qquad \sigma \longrightarrow \widetilde{\phi}^4(\rho) + \sigma = \tilde{\sigma}\,. \tag{3.9}$$

Spelling out

$$g_{ij}(\phi) = \frac{\partial\widetilde{\phi}^k}{\partial\phi^i}\frac{\partial\widetilde{\phi}^l}{\partial\phi^j}\,\widetilde{g}_{kl}(\widetilde{\phi})\,,$$

and solving for $\widetilde{\phi}^3(\rho)$, $\widetilde{\phi}^4(\rho)$ one obtains

$$\widetilde{h}(\widetilde{\phi}^3(\rho)) = h(\rho)\,, \tag{3.10a}$$

$$\int^{\widetilde{\phi}^3(\rho)} \frac{du}{u}\widetilde{b}(u) = \int^{\rho} \frac{du}{u} b(u)\,, \tag{3.10b}$$

$$\widetilde{\phi}^4(\rho) = \int^{\rho} \frac{du}{2u}\frac{a(u)}{b(u)}\left[1 - \frac{\widetilde{a}(\widetilde{\phi}^3(u))}{a(u)}\left(\frac{b(u)}{\widetilde{b}(\widetilde{\phi}^3(u))}\right)^2\right]. \tag{3.10c}$$

These relations can be utilized in several ways. One can use (3.10a) to bring $h(\rho)$ into a prescribed form; then $\widetilde{\phi}^3(\rho)$ is fixed and can no longer help to simplify $b(\rho)$. Alternatively one can use (3.10b) to bring $b(\rho)$ into a prescribed form, in which case $\widetilde{\phi}^3(\rho)$ is fixed by this requirement and can no longer be used to simplify $h(\rho)$. In both cases $a(\rho)$ can largely be changed at will by means of (3.10c). We shall adopt the second option and adjust $b(\rho)$ to be a nonzero constant b. Likewise $a(\rho)$ is adjusted to be some constant a. Summarizing, in an adapted coordinate system the generic form of the target space metric compatible with the above (conformal) Killing vectors is

$$g_{ij}(\phi) = \frac{h(\rho)}{\rho}\,\widehat{g}_{ij}(\phi)\,, \tag{3.11}$$

where $\widehat{g}_{ij}(\phi)$ is given by (3.1) with $b \neq 0$, $a \in \mathbb{R}$. The main modification compared to the initial situation is the ρ-dependent scale factor $h(\rho)/\rho$. The corresponding lagrangian is (2.6) with constant a and $b \neq 0$.

Even within the class of metrics (3.11) some residual transformations (3.9) are possible. Let $\widetilde{h}(\widetilde{\rho})$ and $\widetilde{a}, \widetilde{b} \in \mathbb{R}$ parameterize a metric of this form. Then (3.9) with

$$\widetilde{\phi}^3(\rho) = \rho^p\,, \qquad p := \frac{b}{\widetilde{b}}\,, \qquad \widetilde{\phi}^4(\rho) = \frac{1}{2b}[a - \widetilde{a}p^2]\ln\rho \tag{3.12}$$

maps it onto a metric of the same form, with constants a, b and $h(\rho) = \widetilde{h}(\rho^p)$. In particular one can map $\tilde{h}(\tilde{\rho}) = \tilde{\rho}^{\tilde{p}}$ with $\tilde{p} > 0$ onto $h(\rho) = \rho^{p\tilde{p}}$, where the new power may be negative. Qualitatively this exchanges the role of small and large ρ in the asymptotics of $h(\rho)$.

It is instructive to convert the above target space symmetries into current identities. To this end consider first a generic infinitesimal diffeomorphism $\phi^j \to \phi^j + v^j(\phi)$, $g_{ij} \to g_{ij} - \mathcal{L}_v g_{ij}$. The invariance of $L = \frac{1}{2\lambda}g_{ij}(\phi)\partial\phi^i\partial\phi^j$ can be expressed as

$$\partial^\mu\left[\frac{1}{\lambda}g_{ij}(\phi)v^i(\phi)\partial_\mu\phi^j\right] - \frac{1}{2\lambda}\mathcal{L}_v g_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j + \frac{\delta S}{\delta\phi^i}v^i(\phi) = 0\,. \tag{3.13}$$

Here

$$\mathcal{L}_v g_{ij} = v^m \frac{\partial}{\partial \phi^m} g_{ij} + \frac{\partial v^m}{\partial \phi^i} g_{jm} + \frac{\partial v^m}{\partial \phi^j} g_{im} \,, \tag{3.14}$$

is the Lie derivative with respect to the vector field $v^m(\phi)$. The quantity $\lambda J_\mu(v) = g_{ij}(\phi) v^i(\phi) \partial_\mu \phi^j$ may be interpreted as a "diffeomorphism current". If $v^i(\phi)\partial_i$ is a Killing vector of $g_{ij}(\phi)$ the identity (3.13) simply expresses the conservation of the associated Noether current; so for our (3.11) there are four Noether currents, associated with $\mathbf{e}, \mathbf{h}, \mathbf{f}$, and $\mathbf{t}_-$. The latter reads

$$J_\mu(\mathbf{t}_-) = \frac{b}{\lambda} \partial_\mu \left(\int^\rho \frac{du}{u} h(u) \right) = \frac{b}{\lambda} h \partial_\mu \ln \rho \,. \tag{3.15}$$

For the $\mathrm{O}(1,2)$ Noether currents often the vector basis is more convenient

$$J^i_\mu = \frac{\epsilon}{\lambda} h (n \times \partial_\mu n)^i \,,$$

where

$$J^0_\mu = \frac{\epsilon}{2}[J_\mu(\mathbf{f}) - J_\mu(\mathbf{e})] \,, \qquad J^1_\mu = \frac{\epsilon}{2} J_\mu(\mathbf{h}) \,, \qquad J^2_\mu = \frac{\epsilon}{2}[J_\mu(\mathbf{f}) + J_\mu(\mathbf{e})] \,. \tag{3.16}$$

More interestingly also the two conformal Killing vectors give rise to the on-shell identities:

$$\begin{aligned} \partial^\mu J_\mu(\mathbf{t}_+) &= \rho \partial_\rho \ln h \cdot L \,, & J_\mu(\mathbf{t}_+) &= \frac{b}{2\lambda} h(\rho) \partial_\mu \left(2\sigma + \frac{a}{b} \ln \rho \right) , \\ \partial^\mu J_\mu(\mathbf{d}) &= -\ln \rho \cdot \rho \partial_\rho \ln h \cdot L \,, & J_\mu(\mathbf{d}) &= \frac{b}{\lambda} h(\rho) (\sigma \partial_\mu \ln \rho - \ln \rho \partial_\mu \sigma) \,. \end{aligned} \tag{3.17}$$

Observe that there are only two choices for $h(\rho)$ for which the lagrangian is a total divergence on-shell: $h(\rho) \sim \rho^p$ and $h(\rho) \sim \ln \rho$. (Since $\mathbf{t}_+$ and $\mathbf{d}$ are the only conformal Killing vectors of the target space metric there can be no other identities of this form.) The case $h(\rho) \sim \rho^p$ illustrates that even after the conformal gauge in (2.6) has been chosen, the system still 'remembers' the link to the 2D diffeomorphism invariance. (Recall that in a diffeomorphism invariant theory the lagrangian can always be written as a total divergence on-shell.) In the quantum theory the identities (3.17) can be converted into Ward identities which help to characterize the quantum theory.

3.2 Background field expansion and non-renormalization of ξ^3

The covariant background field method which we shall employ in the quantum theory involves decomposing the fields $\phi = (\Delta, B, \rho, \sigma)$ into a classical background field configuration φ and a formal power series in the quantum fields ξ whose coefficients are functions of φ. The series is defined in terms of the geodesic curve $[0,1] \ni s \to \gamma^j(s)$ from the point $\varphi = \gamma(0)$ to the (nearby) point $\phi = \gamma(1)$, where $\xi^j = \frac{d}{ds}\gamma^j(s)|_{s=0}$ is the tangent vector at φ. E.g. to second order in ξ one has $\phi^j = \varphi^j + \xi^j - \frac{1}{2}\Gamma(\varphi)^j{}_{kl}\xi^k\xi^l + O(\xi^3)$, where $\Gamma(\varphi)^j{}_{kl}$ is the metric connection evaluated at the point φ in target space, i.e. at the background field configuration. Generally we shall write $\phi^j(\varphi;\xi)$ for this series, and refer to ϕ, φ, and ξ as the full field, the background field, and the quantum field, respectively. For our target space metric (3.11) no major simplification occurs with one important exception: the geodesic equation for the 3-component $\gamma^3(s)$ decouples from the others and can be solved in closed form. The solution emanating from φ with tangent vector ξ reads

$$\xi^3 \frac{h(\varphi^3)}{\varphi^3} s = \int_{\varphi^3}^{\gamma^3(s)} \frac{du}{u} h(u) \,. \tag{3.18}$$

In particular $\phi^3 = \rho = \gamma^3(1)$ depends only on φ^3 and ξ^3, the first terms being $\rho = \varphi^3 + \xi^3 + [1/\varphi^3 - \partial_3 \ln h(\varphi^3)](\xi^3)^2 + O((\xi^3)^3)$. This feature turns out to lead to a crucial simplification in the renormalization analysis.

We refer to appendix B for an outline of the renormalization of generic riemannian sigma-models. In our quantum theory eq. (3.11) gives the renormalized target space metric from which the g-dependent counter tensors in (B.2) are computed. In addition to these coupling/source renormalizations also the quantum fields ξ^j are renormalized in a nontrivial way. The transition from the bare fields ξ^j_B to the renormalized ones ξ^j is governed by eq. (B.8). For the target space geometry (3.11) this is still true, with the notable exception of ξ^3:

$$\xi^3_B = \xi^3 \quad \text{to all loop orders.} \tag{3.19}$$

This arises through the combination of the following facts: (i) From (3.18) we know that the inverse normal coordinate expansion for ξ^3 depends only on φ^3 and $\phi^3 - \varphi^3$, i.e. $\xi^3 = \xi^3(\varphi^3; \phi^3 - \varphi^3)$. (ii) The operator $Z(g) - 1$ from which the renormalization $\xi_B(\xi)$ is computed is a scalar differential operator without constant terms built from the covariant derivative ∇_i and the curvature tensors of g_{ij}, both referring to the full field ϕ^j. (iii) A covariant tensor $z_{i_1 \dots i_n}(g)$ of arbitrary rank built from curvature tensors and their covariant derivatives vanishes if $i_k = 4$ for one or more $k = 1, \dots, n$. This can be seen to be a consequence of the flatness of the lower 2×2 block of the target space metric (3.11).

The verification of (3.19) then is straightforward. By (ii) a typical monomial in $Z(g) - 1$ is of the form $z^{i_1 \dots i_n}(g) \nabla_{i_1} \cdots \nabla_{i_n}$, with $n \geq 2$. By (i) it acts on a function of $\rho = \phi^3$ only. One easily checks that then only the $z^{3 \dots 3}(g) \nabla_3^n$ term contributes. However on account of (iii) this vanishes, which proves (3.19).

The importance of (3.19) lies in the fact that composite operators H that are arbitrary functions of (the renormalized full field) $\rho = \phi^3$ do *not* require renormalization, i.e.

$$[\![H(\rho)]\!] = \mu^{d-2} H(\rho) \,, \tag{3.20}$$

where $[\![\,\cdot\,]\!]$ is the normal product defined in eq. (B.12). Hence, up to the trivial μ-prefactor the function H and the composite operator can be identified.

Of course the same is not true for functions depending solely on one of the other fields Δ, B or σ. In particular the associated quantum fields ξ^1, ξ^2 and ξ^4 are renormalized in a nontrivial way. For example, taking advantage of (3.19) one finds from (B.9)

$$\xi^i_B = \xi^i \left[1 + \frac{1}{2-d} \left(\frac{\lambda}{2\pi} \frac{\epsilon}{3h(\rho)} + O(\lambda^2) \right) + \cdots \right], \quad i = 1, 2 \,, \tag{3.21}$$

and a similar more complicated expression for ξ^4_B.

3.3 Structure of the metric counter terms

The (conformal) Killing vectors studied in section 3.1 also constrain the form of the coupling/source counter tensors in (B.2). Here we consider specifically the metric counter tensors $T_{ij}(g)$ since they will be of immediate importance. Similar arguments however apply to the other purely g-dependent counter terms $Z^V(g), N(g), Z(g)$ and $\Psi(g)$. We shall discuss them separately when needed. Here recall the notation $T_{ij}^{(\nu,l)}(g)$, $l \geq \nu$, for the l-loop ν-th order pole term in the metric counter term. For the moment we only need two generic properties of them. First, they are built from the curvature tensors of g_{ij} and their covariant derivatives. Second they can be chosen to transform as as $T_{ij}^{(\nu,l)}(\Lambda^{-1}g) = \Lambda^{l-1}T_{ij}^{(\nu,l)}(g)$, $\Lambda \in \mathbb{R}$, under constant rescalings of the metric.

The first property implies that the geometry of the hyperboloid H_ϵ remains intact. Further — related to the flatness of the lower 2×2 block in (3.11) — all covariant 4-components of the Riemann tensor and its covariant derivatives vanish. A detailed analysis shows that the counter tensors must then be of the form $T_{ij}^{(\nu,l)}(g) \sim \mathrm{diag}(\epsilon/\Delta^2, 1/\Delta^2, *, 0)$, where both the proportionality factor and the $i=j=3$ component are functions of ρ only. The latter are further constrained by the scaling property of the counter tensors. Combined with (3.5) it poses the condition

$$T_{ij}^{(\nu,l)}(\widehat{g})d\phi^i d\phi^j\Big|_{\phi^i\to\exp(-\ln\Lambda\,\mathbf{t}_+)\phi^i} = \Lambda^{l-1}T_{ij}^{(\nu,l)}(\widehat{g})d\phi^i d\phi^j\,, \tag{3.22}$$

on the counter tensors computed from $\widehat{g}_{ij}(\varphi)$. They must therefore be of the form $\rho^{-l}\,\mathrm{diag}(\epsilon\rho/\Delta^2, \rho/\Delta^2, a_l/\rho, 0)$ for some constants a_l. (This structure was already noted in [37].) For a generic $h(\rho)$ in (3.11) only the $i=j=3$ component of the counter tensor can contain ρ-derivatives, so that the before mentioned proportionality factor must be $\zeta_l\, h^{1-l}$, for some number ζ_l.

Later on we need only the $\nu=1$ counter tensors, for which we also have information about the ζ_l through the flat space $\mathrm{O}(1,2)$ sigma-model. In summary we conclude that to all loop orders the counter tensors $T_{ij}^{(1,l)}(g)$ are constrained to be of the form

$$T_{ij}^{(1,l)}(g) = \frac{1}{h^{l-1}}\,\mathrm{diag}\left(\frac{\epsilon\,\zeta_l}{\Delta^2}, \frac{\zeta_l}{\Delta^2}, \frac{S_l(h)}{\rho^2}, 0\right), \qquad \forall\, l\geq 1\,, \tag{3.23}$$

where $S_l(h)$ is a differential polynomial in h that is invariant under constant rescalings of h and vanishes for constant h. The numbers ζ_l are the counter term coefficients of the flat space hyperbolic sigma-model with 2D target space H_ϵ computed (for $l>2$) in the minimal subtraction scheme. Using (B.5) and (3.11) the first three are explicitly given by:

$$\begin{aligned}
T_{ij}^{(1,1)}(g) \;&:\quad \zeta_1 = -\epsilon\,, \qquad S_1(h) = -(\rho\partial_\rho)^2\ln h + \frac{1}{2}(\rho\partial_\rho\ln h)^2\,,\\
T_{ij}^{(1,2)}(g) \;&:\quad \zeta_2 = \frac{1}{2}\,, \qquad S_2(h) = 0\,,\\
T_{ij}^{(1,3)}(g) \;&:\quad \zeta_3 = -\epsilon\frac{5}{12}\,, \quad S_3(h) = -\frac{1}{4}(\rho\partial_\rho)^2\ln h + \frac{1}{12}(\rho\partial_\rho\ln h)^2\,.
\end{aligned} \tag{3.24}$$

Modulo signs the coefficients $l\zeta_l$ indeed coincide with the known beta function coefficients

JHEP12(2002)066

in the O(3) nonlinear sigma model computed in minimal subtraction [42, 43, 44, 45]. The sign pattern is induced by the combination of the overall sign in the lagrangian and the signature of the hyperboloid H_ϵ.

The form of the metric counter tensors in the Ernst-like systems thus is highly constrained, which will be important later on. Nevertheless one sees that the counter terms (3.23) are *not* of the form (3.11) but differ by inverse powers of $h(\rho)$. One might still hope to achieve field theoretical renormalizability by: (i) allowing for non-linear field redefinitions, (ii) promoting the constants a, b (and possibly the coefficient of a $\partial^\mu\sigma\partial_\mu\sigma$ term) to couplings that might get renormalized, and (iii) allowing the renormalized target space metric to be of the generic form (3.11). We state without proof that (i) is indeed enough to absorb the one-loop counter term, but beyond one loop even the combination of (i),(ii) and (iii) is not sufficient to ensure field theoretical renormalizability.

4. Renormalization by preserving the Killing symmetries

Anticipating these lessons we formulate in the following a renormalization procedure that is only slightly weaker than quantum field theoretical renormalizability. For want of a better term we shall refer to it as *conformal renormalizability.* The term is meant to indicate that although the functional form of the lagrangian cannot be preserved identically in the renormalization process, it can be maintained up a a field dependent conformal factor in a way that preserves all, in particular the *conformal* Killing vectors of the original target space metric. We show this in the framework described above to all orders in the loop exapansion and borrow again various results from the renormalization of riemannian sigma-models. To get started recall that in this setting field theoretical renormalizability amounts to the condition that the bare and the renormalized target space metric have the same functional form: $g^B_{ij}(\,\cdot\,) = g_{ij}(\,\cdot\,)$. Since this cannot be achieved for the Ernst-like systems we relax the condition as follows.

4.1 Conformal renormalizability

Motivated by the form of the counter tensors we allow for a change of the target space metric by a singular ρ-dependent prefactor. Explicitly we assume the transition to be of the form

$$g^B_{ij}(\phi) = \mu^{d-2}\, g_{ij}(\phi)\left[1 + \frac{1}{2-d}\sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l H_l(\rho) + \cdots\right], \tag{4.1}$$

with the functions $H_l(\rho)$ to be adjusted. The arguments on *both* sides of (4.1) are the renormalized fields and the dots indicate higher pole contributions. Here we anticipate that the parameters a, b do not get renormalized. The same holds for the coupling λ which therefore merely plays the role of a loop counting parameter. The renormalized metric $g_{ij}(\phi)$ is of the form (3.11), where the function h is part of the specification of the quantum theory. We shall return to the issue of adequately fixing h later. Technically it is advantageous to leave h unspecified and to formulate the renormalization procedure for generic h.

JHEP12(2002)066

In doing so we assume that the bare and the renormalized fields are related by

$$\phi^j_B = \phi^j + \frac{1}{2-d}\sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l \phi^j_l(\phi) + \cdots . \tag{4.2}$$

The only requirement on the functions $\phi^j_l(\phi)$ is that they do not contain derivatives of the fields ϕ^j. In a generic riemannian sigma-model the bare metric value is expanded in terms of the renormalized value as

$$g^B_{ij}(\phi_B) = \mu^{d-2}\left[g_{ij}(\phi) + \frac{1}{2-d}\sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l T^{(1,l)}_{ij}(g) + \cdots\right], \tag{4.3}$$

where we only displayed the counter tensors for the simple poles; cf. eqs. (B.2), (B.3). Combining (4.1), (4.2) and (4.3) one arrives at the finiteness conditions

$$\mathcal{L}_{\phi_l} g_{ij} + H_l(\rho) g_{ij} = T^{(1,l)}_{ij}(g)\,, \qquad l \geq 1\,, \tag{4.4}$$

where $\mathcal{L}_v g_{ij}$ is the Lie derivative (3.14), $T^{(1,l)}_{ij}(g)$ are the counter terms in (3.23), and all quantities are evaluated at the renormalized fields. The ρ-dependence in the $H_l(\rho)$ term marks the difference to a renormalization in the quantum field theoretical sense, see e.g. [47]. The structure (3.23) of the counter tensors implies that $H_l(\rho)$ and $\phi^j_l(\rho)$ scale as

$$H_l(\rho) \longrightarrow \Lambda^{-l} H_l(\rho)\,, \qquad \phi^j_l(\rho) \longrightarrow \Lambda^{-l}\phi^j_l(\rho)\,, \tag{4.5}$$

under $h(\rho) \to \Lambda h(\rho)$.

The finiteness condition (4.4) is easily seen to imply that the fields Δ, B are at most multiplicatively renormalized. For $\phi^3_l(\phi)$ and $\phi^4_l(\phi)$ one has to allow for a non-trivial ρ-dependence and this also turns out to be sufficient. Thus we assume $\phi^3_l(\phi) = \phi^3_l(\rho)$ and $\phi^4_l(\phi) = \phi^4_l(\rho)$. The finiteness condition then amounts to a pair of coupled differential equations for $\phi^3_l(\rho)$ and $H_l(\rho)$. The solutions are

$$\phi^3_l(\rho) = -\zeta_l\,\rho\int_{\rho_l}^{\rho}\frac{du}{u}\frac{1}{h(u)^l}\,, \qquad H_l(\rho) = -\frac{1}{h}\rho\partial_\rho\left(\frac{h\phi^3_l}{\rho}\right), \quad \forall l \geq 1\,. \tag{4.6}$$

The integration constants ρ_l are fixed by requiring that $\phi^3_l(\rho)$ does not contain a term linear in ρ, i.e. $\rho_l = \infty$. One reason why this is a natural choice is that the coefficient of the linear term always has to match that in a renormalization of λ, and both are redundant in the sense that they are not useful for the absorption of counter terms. We thus set both to zero which leads to the above criterion for fixing ρ_l. The same criterion will be recovered below from another viewpoint. With (4.6) known the solutions for $\phi^4_l(\rho)$ can be obtained by a simple integration and read

$$\phi^4_l(\rho) = -\frac{a}{2b\rho}\phi^3_l(\rho) + \frac{1}{2b}\int_{\rho_l}^{\rho}\frac{du}{u}\frac{S_l(h)(u)}{h(u)^l} + d_l\,. \tag{4.7}$$

Again integration constants $d_l = \phi^4_l(\rho_l)$, appear which can be put to zero without loss of generality in the following sense.

The point to observe is that whenever $\phi_l^j(\rho)$ contains an additive contribution proportional to a conformal Killing vector $\mathbf{v}^j$ of $g_{ij}(\phi)$, one can trade it for an additive contribution $\frac{1}{2}\nabla_j \mathbf{v}^j$ to $H_l(\rho)$. In other words the solution of the finiteness condition (4.4) contains an ambiguity in that certain pieces can be shuffled from the Lie derivative term to the term proportional to the metric. The ambiguity is linked to and parameterized by the conformal Killing vectors of g_{ij}. The conformal Killing vector $\mathbf{t}_+$ in the $(\Delta, B, \rho, \sigma)$ coordinates is a linear combination of $(0,0,\rho,0)$ and the Killing vector $(0,0,0,1)$ generating translations in σ. Clearly the ambiguities induced by such linear combinations via the above mechanism just correspond to the arbitrariness in the integration constants in $\phi_l^3(\rho)$, $\phi_l^4(\rho)$. Effectively the above criterion to fix the integration constants thus amounts to removing any part in ϕ_l^j proportional to the $\mathbf{t}_+$ conformal Killing vector. After allowing ϕ_l^4 to also depend linearly on σ the conformal Killing vector $\mathbf{d}$ could similarly be used to add multiples of $\ln\rho \cdot \rho\partial_\rho \ln h$ to $H_l(\rho)$.

This completes the solution of the finiteness condition (4.4). The crucial renormalization is that of the scale factor in (4.1) where, subject to the above specifications, the $H_l(\rho)$ are uniquely determined functionals of h. Since the residues of the higher order poles are determined by those of the first order poles this structure will carry over to the entire divergent part of (4.1), (4.2), and (4.3). Together these renormalizations guarantee the existence of a well-defined renormalized action. In section 4.3 we describe how this extends to the renormalization of correlation functions.

Before taking this up let us evaluate the $l = 1, 2$ renormalization functions $H_l(\rho)$ and $\phi_l^3(\rho)$ for the Ernst system proper, where $h(\rho) = \rho$. One gets $H_1(\rho) = 0$, $H_2(\rho) = 1/(4\rho^2)$ and $\phi_1^3(\rho) = -\epsilon$, $\phi_2^3(\rho) = 1/(4\rho)$. In particular this means the Ernst system is renormalizable in the conventional sense at the 1-loop level but beyond that only in the above 'conformal' sense. More generally one finds

$$H_1(\rho) \equiv 0 \qquad \text{iff} \qquad h(\rho) \sim \rho^p \qquad \text{with} \begin{cases} p > 0 \text{ and } \rho_1 = \infty\,, \\ p < 0 \text{ and } \rho_1 = 0\,. \end{cases} \tag{4.8}$$

Thus, also if h is a generic power of ρ the system remains strictly renormalizable at the 1-loop level. One can also verify that apart from the constant there are no other h functions with that property. The $p > 0$ and the $p < 0$ sector can in principle be related by a field redefinition of the form (3.12) with $b/\tilde{b} < 0$. Both sectors turn out to be equivalent; we consider the $p > 0$ sector throughout.

A bonus of $h(\rho) \sim \rho^p$ is that the 1-loop field renormalizations are gradients of a 'potential'. For $p > 0$ one finds

$$\begin{aligned} \phi_1^j &= \left(0, 0, -\frac{\epsilon}{p}\rho^{-p+1}, \frac{2a\epsilon - p^2}{4bp}\rho^{-p} + d_1\right)\,, \\ \phi_1^j &= -\partial^j \Phi_1\,, \qquad \Phi_1(\rho,\sigma) = \frac{1}{4p}[2\epsilon a + p^2]\,\ln\rho - \frac{bd_1}{p}\rho^p + \frac{\epsilon b}{p}\sigma\,, \end{aligned} \tag{4.9}$$

where d_1 is the integration constant entering through the solution of (4.7) and in Φ_1 an irrelevant additive constant has been omitted. The constant d_1 corresponds to the before mentioned ambiguity in the solution of the finiteness condition associated with the Killing vector $\mathbf{t}_- = \partial_\sigma$, and can be set to zero.

Eq. (4.9) is also convenient to discuss the relation to the observation of de Wit et al [36] that the 1-loop counter term in the Ernst system is a total divergence on-shell (with respect to the base space). In view of the diffeomorphism identity (3.13) this is equivalent to $R_{ij} = \mathcal{L}_{\phi_1} g_{ij}$, for some field renormalization vector $\phi_1^j(\phi)$. Since $H_1(\rho) = 0$ for $h(\rho) = \rho$ this of course is in agreement with our result. If the 4D target space was compact one could also infer from a general theorem by Bourguinion (reviewed in [64]) that ϕ_1^j is the gradient of a scalar without actually computing it. In the case at hand the target space is non-compact (with non-zero curvature) and the fact that ϕ_1^j nevertheless comes out to be the gradient of a scalar is non-trivial. It is also crucial for the physics of the system in that $\phi_1^j \sim \partial^j \Phi_1$ is a necessary condition for conformal invariance [54]. We should also mention that a class of riemannian sigma-models with a target space of Minkowski signature and a null Killing vector has been studied by Tseytlin [38, 39]. However both the setting and the results are not directly related to ours.

4.2 Essential couplings through finite quantum deformations

So far $h(\rho)$ has been treated as λ-independent. Motivated by the analogy to a generalized coupling we now allow it to be of the form[3]

$$h(\rho, \lambda) = h_0(\rho) + \frac{\lambda}{2\pi} h_1(\rho) + \Big(\frac{\lambda}{2\pi}\Big)^2 h_2(\rho) + \cdots . \tag{4.10}$$

As indicated we use $h_0(\rho)$ to denote a λ-independent prefactor in (3.11) and $h(\rho, \lambda)$ for one of the form (4.10). Here (the renormalized=bare) λ serves as the loop counting parameter. The 'adjustable' functions $h_l(\rho) = h_l[h_0]$, $l \geq 1$, are regarded as (local) functionals of h_0. We shall assume $h_0(\rho) \sim \rho^p$, $p > 0$, throughout as this ensures 1-loop renormalizability. Moreover we consider only genuine deformations where not all of the $h_l(\rho)$, $l \geq 1$, are again proportional to ρ^p.

In addition to being technically rather natural the deformation (4.10) has a profound physical significance. In Weinberg's terminology [1] it replaces the "inessential" coupling $h_0(\rho) \sim \rho^p$ by an "essential" coupling function $h(\,\cdot\,, \lambda)$. Roughly speaking an inessential coupling is one whose flow is affected by field redefinitions and which may continue to run even at a fixed point. A simple test is to compute the variation of the lagrangian with respect to the bare quantity. If it comes out a total divergence modulo the equations of motion the quantity is an "inessential coupling". Applied to the Einstein-Hilbert action this criterion disqualifies Newton's constant as an inessential coupling [1]. It is only if one includes a cosmological constant term and/or higher order curvature scalars that the ratios of their prefactors become "essential couplings" — in the space of which one can search for a fixed point [10, 11]. In the context of the 2-Killing reduction the deformation (4.10) achieves precisely the same: since $h\frac{\partial}{\partial h} L = L = -\lambda \frac{\partial}{\partial \lambda} L$ both the reduced Newton constant λ and $h(\,\cdot\,)$ will be inessential if L can be written as a total divergence on-shell. Going back to eqs. (3.17) one sees that this is the case iff $h(\rho) \sim \rho^p$ or $h(\rho) \sim \ln \rho$. Since we insist on 1-loop renormalizability and exclude trivial deformations in (4.10) the set of functions $h(\,\cdot\,, \lambda)$ qualifies as an *essential coupling*. In particular the space of these functions is the

[3] I am indebted to P. Forgács for suggesting this.

appropriate arena to search for a fixed point. We shall take up the search in section 5. Our immediate concern though is to determine the impact the functions h_l, $l \geq 1$, have on the renormalization of the system(s).

We begin by examining the effect of (4.10) on the solution (4.6) of the finiteness condition. In a renormalizable quantum field theory finite coupling redefinitions correspond to a change of scheme. Their impact can be studied simply by substituting into the solution computed in the original scheme and re-expanding in powers of the loop counting parameter. Since in the case at hand non-algebraic manipulations were involved in arriving at (4.6) we made sure that such a substitution procedure is legitimate also here by going back to the finiteness conditions. That is we used (4.10) in the renormalized metric and the counter tensors from the beginning and expanded in powers of λ to arrive at a modified set of finiteness conditions (4.4), which is then solved as before.

The general finiteness condition for the λ-dependent quantities is obtained along the same lines as before, i.e. by combining the general identity

$$\begin{aligned} g^{B}_{ij}(\phi_B)\partial^\mu\phi^i_B\partial_\mu\phi^j_B &= g^{B}_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j + \frac{1}{2-d}\,\mathcal{L}_{\Xi(\phi,\lambda)}g^{B}_{ij}(\phi)\,\partial^\mu\phi^i\partial_\mu\phi^j\,, \\ \phi^j_B = \phi^j + \frac{1}{2-d}\,\Xi^j(\rho,\lambda)\,, &\qquad \Xi^j(\rho,\lambda) := \sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l\phi^j_l(\rho)\,, \end{aligned} \tag{4.11}$$

with the relation between the bare and the renormalized metric functional (4.1) specific for the Ernst system

$$g^{B}_{ij}(\phi) = \mu^{d-2}\,g_{ij}(\phi,\lambda)\left[1 + \frac{1}{2-d}H(\rho,\lambda) + \cdots\right], \qquad H(\rho,\lambda) := \sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l H_l(\rho)\,. \tag{4.12}$$

The renormalized metric here depends on λ through a prefactor $h(\rho,\lambda)$ of the form (4.10). Again the arguments on *both* sides of (4.12) are the renormalized fields. The finiteness condition obtained from (4.11), (4.12) reads

$$\mathcal{L}_{\Xi(\rho,\lambda)}g_{ij}(\phi,\lambda) + H(\rho,\lambda)g_{ij}(\phi,\lambda) = \lambda T^{(1)}_{ij}\left(\frac{g(\phi,\lambda)}{\lambda}\right), \tag{4.13}$$

with $T^{(1)}_{ij}(g)$ from eqs. (3.23), (B.3). If one now expands in powers of λ the $l = 1$ equation coincides with (4.4) but the $l \geq 2$ equations are modified. The solutions $H_l(\rho)$, $\phi^j_l(\rho)$ will depend on $h_0, \dots, h_{l-1}$. We won't need the explicit form of the modified $l \geq 2$ finiteness conditions because eventually their solution turns out to coincide with that of the substitution procedure, despite the fact that non-algebraic manipulations are involved. Strictly speaking this holds only if the integration constants ρ_l are assumed to be equal, otherwise some trivial ambiguities have to be taken into account.

The result can be summarized by saying that simply substituting h for h_0 and re-expanding in powers of λ produces the correct solution of the modified finiteness conditions. For illustration let us quote the explicit three-loop results for the solution of the finiteness condition for $h(\rho,\lambda)$ of the form (4.10): the one loop solutions are unchanged, i.e. are given

JHEP12(2002)066

by (4.6) with $h(\rho) = h_0(\rho)$. The two and three loop coefficients are modified according to

$$\phi_2^3(\rho) = \rho \int_\rho^\infty \frac{du}{u h_0(u)^2}[\zeta_2 - \zeta_1 h_1(u)]\,, \tag{4.14a}$$

$$\phi_3^3(\rho) = \rho \int_\rho^\infty \frac{du}{u h_0(u)^3}[\zeta_3 - 2\zeta_2 h_1(u) - \zeta_1 (h_2 h_0 - h_1^2)(u)]\,, \tag{4.14b}$$

$$H_2(\rho) = -\frac{1}{h_0}\rho\partial_\rho\Big(\frac{h_0\phi_2^3}{\rho}\Big) - \phi_1^3(\rho)\partial_\rho\Big(\frac{h_1}{h_0}\Big)\,, \tag{4.14c}$$

$$H_3(\rho) = -\frac{1}{h_0}\rho\partial_\rho\Big(\frac{h_0\phi_3^3}{\rho}\Big) - \phi_2^3(\rho)\partial_\rho\Big(\frac{h_1}{h_0}\Big) + \phi_1^3(\rho)\partial_\rho\left\{\frac{1}{2}\Big(\frac{h_1}{h_0}\Big)^2 - \frac{h_2}{h_0}\right\}. \tag{4.14d}$$

Here we took $\rho_l = \infty$ for all the integration constants. Observe that the new solutions obey the scaling (4.5) if h_l is assigned scaling dimension $1-l$. For the existence of the integrals in (4.14a,b) only a mild constraint on the large ρ asymptotics is needed, for example

$$\frac{h_l}{h_0^{l+1}} = o\Big(\frac{1}{\ln^{1+p}\rho}\Big)\,, \quad p>0 \qquad \text{for } \rho\to\infty\,, \quad l\geq 1\,, \tag{4.15}$$

with $1/h_0 \sim o(1/\ln^{1+p}\rho)$ is sufficient. As stated earlier we specifically take $h_0(\rho)\sim\rho^p$, $p>0$, throughout as this ensures 1-loop renormalizability. We also assume that derivatives are well-behaved, e.g. $\partial_\rho h = O(h)$, for large ρ, etc.

Having justified the the substitution procedure we can use (4.6) to obtain closed expressions for $H(\rho,\lambda)$ and $\Xi^j(\rho,\lambda)$ valid also for $h(\rho,\lambda)$ of the form (4.10). The counter terms can be written as

$$\lambda T_{ij}^{(1)}(g/\lambda) = \mathrm{diag}\left(\frac{\epsilon h}{\Delta^2}B_\lambda\Big(\frac{\lambda}{h}\Big),\ \frac{h}{\Delta^2}B_\lambda\Big(\frac{\lambda}{h}\Big),\ \frac{h}{\rho^2}S(\rho,\lambda),\ 0\right), \tag{4.16}$$

where $S(\rho,\lambda) = \sum_{l\geq1}(\frac{\lambda}{2\pi})^l h^{-l}S_l(h)$. Further

$$B_\lambda(\lambda) = \sum_{l\geq1}\zeta_l\Big(\frac{\lambda}{2\pi}\Big)^l = \int_0^\lambda \frac{ds}{s^2}\beta_\lambda(s)\,, \qquad \beta_\lambda(\lambda) = \lambda^2\frac{\partial}{\partial\lambda}\sum_{l\geq1}\zeta_l\Big(\frac{\lambda}{2\pi}\Big)^l\,, \tag{4.17}$$

is related to the beta function of the flat space O(1,2) sigma-model with target space H_ϵ, computed in the minimal subtraction scheme. The solutions of (4.13) come out as

$$\begin{aligned}
H(\rho,\lambda) &= -\frac{1}{h(\rho,\lambda)}\rho\partial_\rho\left[h(\rho,\lambda)\frac{\Xi^3(\rho,\lambda)}{\rho}\right].\\
\Xi^3(\rho,\lambda) &= \rho\int_\rho^\infty \frac{du}{u}B_\lambda\left(\frac{\lambda}{h(u,\lambda)}\right),\\
\Xi^4(\rho,\lambda) &= -\frac{a}{2b\rho}\Xi^3(\rho,\lambda) + \frac{1}{2b}\int^\rho \frac{du}{u}S(u,\lambda)\,,
\end{aligned} \tag{4.18}$$

where again $\rho_l = \infty$, $l\geq 1$, was assumed, and in Ξ^4 a λ-dependent integration constant was absorbed into the lower integration boundary. These expressions generalize (4.6), (4.7).

The condition (4.15) also implies that $h(\rho,\lambda)$ grows for $\rho\to\infty$ at least like ρ^p, $p>0$, though off-hand it could grow much faster. The counter term tensor (4.16) thus has a finite and universal limit

$$\lambda T^{(1)}_{ij}(g/\lambda) \;\longrightarrow\; \frac{\lambda}{2\pi}\,\mathrm{diag}\left(\epsilon\frac{\zeta_1}{\Delta^2}, \frac{\zeta_1}{\Delta^2}\frac{S_1(\rho)}{\rho^2}, 0\right) \qquad \text{for } \rho\to\infty\,, \tag{4.19}$$

where the subleading terms are down by a power of $1/h$. Geometrically $R(g) = -2\epsilon/h(\rho)$ is the scalar curvature of the metric (3.11) so that the limit (4.19) corresponds to weak curvature. (For $p<0$ the relevant limit would be $\rho\to 0$.) Since (4.19) coincides with the 1-loop counter term one can read off the asymptotic solution H_∞, Ξ^j_∞ of the finiteness condition from the 1-loop results. There is no reason to introduce an ad-hoc λ-dependence into the asymptotics $h_\infty(\rho)$ of $h(\rho)$. Assuming it to be λ-independent enforces $h_\infty(\rho) = h_0(\rho)\sim\rho^p$ and the solution triggered by (4.19) is

$$H_\infty = 0\,, \qquad \Xi^j_\infty = \frac{\lambda}{2\pi}\phi^j_1\,, \tag{4.20}$$

with ϕ^j_1 given by (4.9).

4.3 Renormalized currents

In the previous sections we have been concerned with the renormalization of the basic lagrangian. Of course eventually one is interested in constructing correlation functions of suitable composite operators, where the lagrangian (co-)determines the perturbative measure. Since for the Ernst-like systems (2.10) the classical observables are built from the Noether currents and h it seems natural to primarily aim at renormalizing their correlation functions. In addition the constraints ought to be constructed as composite operators. This can be achieved by including suitable *local* sources in the lagrangian such that after renormalization the composite operators can be obtained by functional differentiation. The sources of course are likewise subject to renormalization and the problem consists in showing that they can be included in a way that preserves the "conformal renormalizability" of the system in the sense introduced above. Technically it is again convenient to borrow results from the renormalization of riemannian sigma-models. We shall use the results and the notations of appendix B throughout.

In a first step we determine the analogue of the source-extended lagrangian (B.1) appropriate for the Ernst-like systems. In the purely metric part a local source can be included by making $h(\rho)$ explicitly x-dependent: $h(\rho)\to h(\rho;x)$. This manifestly preserves the (conformal) symmetries of the target space metric (3.11), in particular $\overline{\partial}_\mu g_{ij} = \overline{\partial}_\mu \ln h\, g_{ij}$. The vector sources should evidently respect the $\mathrm{O}(1,2)$ symmetry. For the moment we are only interested in renormalizing the four Noether currents (3.16) and (3.15) possibly multiplied by functions of ρ. We thus introduce an $\mathrm{O}(1,2)$ vector source $\omega^j_\mu(\rho;x)$, $j=0,1,2$, and a scalar source $\omega_\mu(\rho;x)$. As indicated both are vectors on the base space, functions of x and functionals of ρ. The corresponding source term is $\omega_\nu\cdot J_\mu + \omega_\nu J_\mu(\mathbf{t}_-)$ and replaces $V_{\nu i}\partial_\mu\phi^i$ in (B.1). Since we describe the Ernst-like systems in terms of an action on a flat base space there is in principle no need to minimally couple the system to an external background metric $\widehat{\gamma}_{\mu\nu}$. In practice though a source term $R^{(2)}(\widehat{\gamma})\Phi$ is a convenient tool to generate an improvement term for the energy momentum tensor (i.e. to the constraints). Anticipating that such an improvement term is needed later, we minimally couple the original system to a fiducial background metric $\widehat{\gamma}_{\mu\nu}$ on the base space and include a source term $R^{(2)}(\widehat{\gamma})\Phi$. It is not hard to see (cf. section 6) that the interpretation of Φ as a potential for the improvement term constrains it to be of the form $\Phi = f(\rho) + f_0\sigma$, for a constant f_0. Finally F mainly serves as a tool to determine the generalized "wave function renormalizations" and will be chosen accordingly. Including in it a term quadratic in the fields also provides an infrared regulator. In summary we arrive at the following source-extended lagrangian for the Ernst-like systems

$$\begin{aligned}\lambda L(G;\phi) \;=\; & \frac{h(\rho;x)}{2\rho}\widehat{g}_{ij}(\phi)\widehat{\gamma}^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j + \widehat{\gamma}^{\mu\nu}[\omega_\nu\cdot J_\mu + \omega_\nu J_\mu(\mathbf{t}_-)] + \\ & + \frac{1}{2}R^{(2)}(\widehat{\gamma})[f(\rho) + f_0\,\sigma] + F(\phi)\,. \end{aligned} \tag{4.21}$$

Here G now stands for the collection of generalized couplings/sources $\{h, \omega_\mu, f, F\}$.

As a simple illustration let us compute the "wave function" renormalization of the spin fields n^j, $j = 0,1,2$, to lowest order. In the present framework they are treated as composite operators $n^j = n^j(\Delta, B)$ according to (A.5). Taking as scalar source $F(\phi) = \omega(\rho;x)l_j n^j(\Delta, B)$, for a singlet $\omega(\rho;x)$ and a constant $\mathrm{O}(1,2)$ vector l_j one finds from (B.2) and eqs. (3.20), (4.2), (4.9)

$$n^j_{\scriptscriptstyle B} = n^j\left[1 - \frac{1}{2-d}\frac{\lambda}{2\pi}\left(\frac{\epsilon}{h(\rho)} + \partial_\rho \ln\omega(\rho)\,\phi_1^3(\rho)\right) + \cdots\right]. \tag{4.22}$$

In the decoupling limit of constant $h(\rho) = 1/\lambda_0$ and constant source ω this correctly reproduces the leading wave function renormalization of the spins in the $\mathrm{O}(1,2)$ sigma-model without coupling to gravity.

Next we consider the renormalization of the Noether currents. In a renormalizable quantum field theory conserved Noether currents are not renormalized in the sense that the coupling and field renormalizations are enough to render them finite and conserved as composite operators. The goal in the following is to derive an analogous result for the Ernst-like systems which are renormalizable only in the broader "conformal" sense. In order to explain the result we write $J^i_\mu(h;\phi)$ for the $\mathrm{O}(1,2)$ Noether currents (3.16) and $J_\mu(h;\rho)$ for the Noether current (3.15) associated with the σ-translations. Then the following "non-renormalization" results hold to all orders in the loop expansion:

$$\begin{aligned}[\![J^i_\mu(h;\phi)]\!] &= J^i_\mu(h_{\scriptscriptstyle B};\phi_{\scriptscriptstyle B})\,, \qquad i = 0,1,2\,,\\ [\![J_\mu(h;\rho)]\!] &= \mu^{d-2}J_\mu(h;\rho)\,. \end{aligned} \tag{4.23}$$

The normal products are defined in eq. (B.12); in general evidently the functional form of a dimension 1 operator will change under renormalization. For the $\mathrm{O}(1,2)$ Noether currents however this is not the case and (4.23) states that they can be rendered finite and conserved as composite operators by renormalizing the fields and the generalized coupling $h(\,\cdot\,)$. The even stronger result for $J_\mu(h;\rho)$ is of course related to (3.20).

JHEP12(2002)066

To show (4.23) we employ the consistency conditions (B.15) entailed by the diffeomorphism Ward identity. Combined with (B.2) the first relation implies $[g_{ij} + T_{ij}(g)]\mathbf{v}^j = Z^V(g)_i^j \mathbf{v}_j$, for a Killing vector $\mathbf{v}^j$. Further, as noted in section 3.3, the Killing vectors are eigenvectors of the metric counter terms $T_{ij}(g)$, in the case of $\mathbf{t}_-$ with vanishing eigenvalue. By the above consistency condition this carries over to $Z^V(g)$:

$$\begin{aligned} Z^V(g)_i^j(\mathbf{t}_-)_j &= (\mathbf{t}_-)_i\,, \\ Z^V(g)_i^j\mathbf{v}_j &= \left[1 + \frac{1}{2-d}B_\lambda\Big(\frac{\lambda}{h}\Big) + \cdots\right]\mathbf{v}_i\,, \quad \mathbf{v} = \mathbf{e},\mathbf{h},\mathbf{f}\,, \end{aligned} \tag{4.24}$$

where (4.16) has been used. The second equation in (B.15) implies for a Killing vector

$$\mathbf{v}_i \cdot \frac{\partial Y}{\partial V_i^\mu} = \mathbf{v}^i Z^V(g)_i^j V_{\mu j} - Z(\mathbf{v}^i V_{\mu i}) + \mathbf{v}^i N_i^{\,jk}(g)\overline{\partial}_\mu g_{jk}\,. \tag{4.25}$$

The last term on the right hand side is readily seen to vanish, for the other two we recall that the sources $V_{\mu i}$ relevant for the Noether currents are proportional to the respective Killing vector. The proportionality factor is a function of ρ and x only and can be pulled out of the differential operators in $Z^V(g)$ and $Z(g)$. In the end it is set to zero and kills the right hand side of (4.25). Thus only the counter terms in (4.24) remain. For $J_\mu(h;\rho)$ this directly gives the second equation in (4.23). The counter terms for the O(1,2) currents are obviously the ones associated with the metric renormalization in (4.3), but they can also be interpreted as a generalized coupling renormalization via

$$h_B(\rho_B) = \mu^{d-2}h(\rho)\left[1 + \frac{1}{2-d}B_\lambda\Big(\frac{\lambda}{h}\Big) + \cdots\right]. \tag{4.26}$$

This follows from eq. (5.1) below, the ρ_B renormalization in (4.11), (4.18), and (4.13). The purely (Δ, B) dependent part of the currents remains unrenormalized and one arrives at the first equation in (4.23).

5. Flow equations

The renormalization until here was performed at a fixed normalization scale μ. Changing the scale leaves the bare quantities unaffected but the renormalized ones have to compensate for it by carrying a μ-dependence. It turns out that both the function $h(\,\cdot\,,\lambda)$ and the fields ρ, σ are subject to nontrivial flow equations. The former is analogous to the running coupling in an ordinary quantum field theory. The latter is induced by the h-dependence of the renormalized fields and generalizes the concept of an anomalous dimension matrix.

5.1 Gravitationally dressed beta function

Recall that the function h in (3.11) could be prescribed at will and constituted part of the specification of the quantum theory. The same is true for $h(\,\cdot\,,\lambda)$ of the generic form (4.10); in order not to clutter the notation we will often suppress the λ-dependence in the following. As μ changes the functional form of h can in general not be maintained. Rather $h(\,\cdot\,)$ has to become a function $\overline{h}(\,\cdot\,,\mu)$ of the normalization scale μ which is analogous to the running coupling in an ordinary field theory. Of course all functions connected by varying μ must be regarded as equivalent and do not define different theories.

A natural way to define a beta function for $h(\rho)$ is to interpret (4.12) as a relation between the bare and the renormalized scale factor

$$h^{B}(\rho) = \mu^{d-2} h(\rho,\lambda) \left[1 + \frac{1}{2-d} H(\rho,\lambda) + \cdots \right], \tag{5.1}$$

allowing now the renormalized h to be of the generic form (4.10). Following the usual procedure yields the beta function

$$\lambda \beta_h(h/\lambda) = (2-d) h(\rho) - h(\rho) \int du\, h(u) \frac{\delta H(\rho,\lambda)}{\delta h(u)}\,, \tag{5.2}$$

where we suppress the λ-dependence of h. Since the functional derivative in (5.2) will frequently reappear we introduce the shorthand

$$\dot{X}(\rho) := \int du\, h(u) \frac{\delta X(\rho)}{\delta h(u)}\,, \tag{5.3}$$

for a functional $X(\rho) = X[h](\rho)$ of $h(\rho)$. Observe that for any differential or integral polynomial X_l in h which is homogeneous of degree l, the functional derivative (5.3) just measures the degree, $\dot{X}_l = l X_l$.[4] In particular the '$\cdot$' derivatives of the solution (4.18) of the finiteness condition will be needed frequently and come out as

$$\begin{aligned}
\dot{H} &= -\frac{1}{h} \rho \partial_\rho \left[\frac{h}{\rho} \dot{\Xi}^3 \right], \\
\dot{\Xi}^3 &= -\rho \int_\rho^\infty \frac{du}{u} \frac{h(u)}{\lambda} \beta_\lambda\left(\frac{\lambda}{h(u)} \right), \\
\dot{\Xi}^4 &= -\frac{a}{2b\rho} \dot{\Xi}^3 + \frac{1}{2b} \int^\rho \frac{du}{u} \dot{S}(u,\lambda)\,.
\end{aligned} \tag{5.4}$$

In $\dot{\Xi}^4$ we absorbed a λ-dependent additive constant into the lower integration boundary and used $\dot{S}(\rho,\lambda) = -\sum_{l\geq 1} (\frac{\lambda}{2\pi})^l l h^{-l} S_l(\rho)$, as $\dot{S}_l(\rho) = 0$. For the $\beta_h(h)$ function (5.2) this yields explicitly

$$\lambda \beta_h(h/\lambda) = (2-d) h - \rho \partial_\rho \left[h \int_\rho^\infty \frac{du}{u} \frac{h(u)}{\lambda} \beta_\lambda\Big(\frac{\lambda}{h(u)} \Big) \right]. \tag{5.5}$$

Interestingly $\beta_h(h)$ comes out to be a total $\ln \rho$-derivative. Further the functional beta function for h is completely determined by the conventional beta function in (4.17) of the O$(1,2)$ sigma-model in flat space, and thus can be viewed as a "gravitationally dressed" version of the latter. A similar concept was (in a somewhat different context and at the 1-loop level) employed in [73], from which we borrow the term. See also [72]. Eq. (5.5) is a structural result valid to all loop orders. The corresponding flow equation is

$$\mu \frac{d}{d\mu} \overline{h} = \lambda \beta_h \Big(\frac{\overline{h}}{\lambda} \Big) \qquad \text{with} \qquad \overline{h}(\rho,\mu_0) = h(\rho)\,. \tag{5.6}$$

[4] As a warning let us add that this simple rule applies only if h is unconstrained. For example in taking the '$\cdot$' derivative of eq. (C.3) one has to take into account that ρ is functionally dependent on h.

Before discussing its properties let us briefly comment on the relation of $\beta_h(h)$ to the tensorial beta function in (B.17). To find the relation one operates with $1 - \int h\frac{\delta}{\delta h}$ on both sides of (4.13). Using $\dot{T}_{ij}^{(1,l)}(g) = (1-l)T_{ij}^{(1,l)}(g)$ this gives

$$(d-2)g_{ij} + \lambda\beta_{ij}\left(\frac{g}{\lambda}\right) = -h\left[h^{-1}\lambda T_{ij}^{(1)}\left(\frac{g}{\lambda}\right)\right]^{\cdot} = -\dot{H}g_{ij} - \mathcal{L}_{\dot{\Xi}}g_{ij}\,. \tag{5.7}$$

One sees that the piece proportional to g_{ij}/h gives back the $\beta_h(h)$ function, while the Lie derivative term is induced by the nonlinear field renormalizations. However a naive transcription of the tensorial flow equation (B.17) would *not* give rise to consistent equations for the individual metric components. This is because in parameterizing the transition between the bare and the renormalized quantities via equations (4.1), (4.2), we also decided to treat h as a generalized coupling on which the nonlinear field renormalizations depend on. For a generic riemannian sigma-model on the other hand only redefinitions (B.23) of fields are considered that are independent of the renormalized metric. Indeed, even if one would accept the odd feature that the coordinates on the target space are co-determined by the metric tensor (and vice versa) it would be impossible to disentangle the combined μ-dependence in $g_{ij}(\phi[g](\mu),\mu)$ with respect to the 'moving' coordinates $\phi^j[g](\mu)$.

Next let us verify that (5.6) gives sensible answers in two simple special cases. The first one is the decoupling limit where h equals a constant. For constant h the $\partial^\mu n\cdot\partial_\mu n$ term in the action based on (3.11) decouples from the $\partial^\mu \ln\rho\,\partial_\mu(\sigma+\frac{1}{2}\ln\rho)$ term. The field redefinitions of the latter do not effect the former and one expects to recover the ordinary beta function for a sigma model with the 2D hyperbolic target space (A.4). This is indeed the case provided (5.6) is interpreted as a flow equation for the proportionality factor. (Recall that for the Ernst-type systems λ does not get renormalized and, at fixed μ, just serves as a loop counting parameter while for constant h the coupling does get renormalized and is conceptually distinct from the loop counting parameter). For $h = \mathrm{const} = 1/\lambda_0$ the $\beta_h(h)$ function of eq. (5.5) evaluates to

$$\beta_h\left(\frac{h}{\lambda}\right)\bigg|_{h=1/\lambda_0} = \frac{1}{(\lambda\lambda_0)^2}\beta_\lambda(\lambda\lambda_0)\,. \tag{5.8}$$

As expected the coupling λ_0 only occurs in the combination $\lambda\lambda_0$ and thus can serve in itself as a loop counting parameter. Putting λ equal to unity the flow equation (5.6) becomes

$$h(\rho) = \frac{1}{\lambda_0}: \qquad \mu\frac{d\overline{\lambda}_0}{d\mu} = -\beta_\lambda(\overline{\lambda}_0) = -\frac{\overline{\lambda}_0^2}{2\pi}\left(-\epsilon + \frac{\overline{\lambda}_0}{2\pi}\right) + \cdots\,, \tag{5.9}$$

which is the correct flow equation for a flat space $\mathrm{O}(1,2)$ sigma-model with target space H_ϵ. As usual the dependence on the renormalized coupling λ_0 enters the solution of (5.9) only through the initial condition $\overline{\lambda}(\mu_0) = \lambda_0$, for some μ_0.

Another special case is the abelian subsector where $B \equiv 0$. This amounts to deleting the second row and column in the target space metric (3.11) upon which its scalar curvature vanishes. Repeating the previous computations one finds that the counter tensors are of the form

$$T_{ij}^{(1,l)}(g) = \mathrm{diag}(0, \rho^{-2}h^{1-l}(\rho)S_l(\rho), 0)\,, \tag{5.10}$$

with

$$S_1(h) = -\frac{1}{2}(\rho\partial_\rho)^2 \ln h + \frac{1}{4}(\rho\partial_\rho \ln h)^2\,, \qquad S_2(h) = S_3(h) = 0\,. \tag{5.11}$$

The solution of the general finiteness condition (4.13) then is

$$\Xi^3(\rho) = C(\lambda)\rho\,, \qquad H(\rho) = -C(\lambda)\rho\partial_\rho \ln h\,, \tag{5.12}$$

for some integration constant $C(\lambda)$, while $\Xi^4(\rho)$ in (4.18) retains its form. The formula (5.2) for the functional beta function still applies and gives $\beta_h(h) \equiv 0$ (at $d = 2$), to all loop orders. This is gratifying because in a reduced phase space quantization the abelian system is non-interacting and can be renormalized simply by normal ordering. The results (5.12) also illustrate again the features discussed after eq. (4.7): $\Xi^3(\rho)$ is proportional to the $\rho\partial_\rho$ conformal Killing vector and $H(\rho)$ is the associated conformal factor. A renormalization of ρ_B is not enforced by the counter terms; putting $C(\lambda) = 0$ gives $\rho_B = \rho$ and $H \equiv 0$. On the other hand σ does get renormalized, although (assuming $S_l(h) = 0$ for all $l > 1$, and putting the integration constants to zero) only by a 1-loop contribution. Taking $h(\rho) = \rho^p$ one has $\phi_1^4(\rho) = -\frac{p}{8b}\rho^{-p}$.

An initially puzzling feature of $\beta_h(h)$ is that it comes a total $\ln\rho$-derivative. Restoring the interpretation of $\rho = \rho(x)$ as a field on the 2D base space, however, it has a natural interpretation: An immediate consequence of (5.5) is that (putting $d = 2$) contour integrals of the form

$$\int_C dx^\mu \partial_\mu \ln\rho\, \overline{h}(\rho,\mu)\,, \tag{5.13}$$

are μ-independent for any closed contour C in the base space. They are thus invariants of the flow and can be used to discriminate the inequivalent quantum theories (redundantly) parameterized by $h_l[h_0]$. With the initial condition $\overline{h}(\rho,\mu_0) = h(\rho)$ the μ-independence of (5.13) is equivalent to $\partial^\mu[(\overline{h} - h)\partial_\mu \ln\rho] = 0$. On the other hand the (classical and quantum) equations of motion for ρ with respect to the h-modified action are just $\partial^\mu(h\partial_\mu \ln\rho) = 0$. Combining both we find that the significance of $\beta_h(h)$ being a total $\ln\rho$-derivative is that this feature preserves the equations of motion for ρ under the μ-evolution of $\overline{h}(\,\cdot\,,\mu)$:

$$\int^\rho \frac{du}{u} h(u) \;\text{harmonic} \quad \Longrightarrow \quad \int^\rho \frac{du}{u}\overline{h}(u,\mu) \;\text{harmonic}\,. \tag{5.14}$$

This provides an important consistency check as (5.14) is also required by the non-renormalization of the ∂_σ Noether current, cf. (4.23).

Finally let us consider the $\rho \to \infty$ limit of the flow equation (5.6). On account of the reasoning leading to eq.s (4.20) one will want the initial $h(\rho) = \overline{h}(\rho,\mu_0)$ to have a λ-independent asymptotics $h_\infty(\rho) \sim \rho^p$. From (4.20) it then follows that $\beta_h(h) \to 0$ for $\rho \to \infty$. For $\rho \to \infty$ the $\overline{h}$-flow therefore freezes: $\overline{h}(\rho,\mu) \sim h_\infty(\rho)$ for all μ. This guarantees that the functional flow is solely driven by the counter terms and not by artifacts. Subject to these asymptotic boundary conditions the $\overline{h}$-flow is unambiguously defined. In the next section we proceed to determine its fixed points and study the linearized flow in its vicinity.

5.2 Fixed point function and linearized flow

The stationary points of the flow (5.6) can be computed by converting the vanishing condition for the $\beta_h(h)$ function (5.5) into a differential equation. It reads

$$\frac{\lambda}{2\pi}\rho\partial_\rho h = C(\lambda)h^2\,\frac{h}{\lambda}\beta_\lambda\left(\frac{\lambda}{h}\right), \tag{5.15}$$

for some $C(\lambda) = \sum_{l\geq 0} C_l(\lambda/2\pi)^l$ with constant C_l. This can be solved recursively for h_0, h_1, etc. We denote the solutions by $h_l^{\rm beta}(\rho)$. The minimal solution corresponding to a λ-independent $C(\lambda) = C_0 = p/\zeta_1$ is

$$h^{\rm beta}(\rho,\lambda) = \rho^p - \frac{\lambda}{2\pi}\frac{2\zeta_2}{\zeta_1} - \left(\frac{\lambda}{2\pi}\right)^2\frac{3\zeta_3}{2\zeta_1}\rho^{-p} + \cdots\,, \tag{5.16}$$

Switching on the constants C_1, C_2, etc produces deformations of the functions $h_l^{\rm beta}(\rho)$. In order to compute them we expand (5.15) in powers of λ to find

$$\begin{aligned}
\rho\partial_\rho h_0 &= C_0\zeta_1 h_0\,,\\
\rho\partial_\rho h_1 &= C_0\zeta_1 h_1 + C_1\zeta_1 h_0 + 2C_0\zeta_2\,,\\
\rho\partial_\rho h_2 &= C_0\zeta_1 h_2 + C_1\zeta_1 h_1 + C_2\zeta_1 h_0 + \frac{3C_0\zeta_3}{h_0} + 2C_1\zeta_2\,,
\end{aligned} \tag{5.17}$$

etc. From them the solutions h_0, h_1, h_2, etc are computed recursively. One finds $h_0(\rho) = \rho^p/\lambda_0$, with $p = \zeta_1 C_0$, and a normalization constant λ_0. Further

$$\begin{aligned}
h_1^{\rm beta} &= -\frac{2\zeta_2}{\zeta_1} + \frac{C_1}{C_0}h_0\ln h_0\,,\\
h_2^{\rm beta} &= -\frac{3\zeta_3}{2\zeta_1}h_0^{-1} + \frac{C_2}{C_0}h_0\ln h_0 + \frac{1}{2}\left(\frac{C_1}{C_0}\right)^2 h_0\ln^2 h_0\,,
\end{aligned} \tag{5.18}$$

where trivial additive terms proportional to h_0 have been omitted. Generally h_l, $l\geq 1$, is a function of h_0 containing l deformation parameters, C_k/C_0, $k = 1,\ldots,l$. The significance of these parameters can be seen from the $\rho\to\infty$ limit, where the curvature radius of the target space manifold approaches zero. In this limit $h^{\rm beta}(\rho,\lambda) = h_\infty(\rho,\lambda)[-\frac{2\zeta_2}{\zeta_1} + O(h_0^{-1})]$, with

$$\frac{h_\infty(\rho,\lambda)}{h_0(\rho)} = 1 + \frac{\lambda}{2\pi}\frac{C_1}{C_0}\ln h_0 + \left(\frac{\lambda}{2\pi}\right)^2\left(\frac{C_2}{C_0}\ln h_0 - \frac{1}{2}\left(\frac{C_1}{C_0}\right)^2\ln^2 h_0\right) + O(\lambda^3)\,. \tag{5.19}$$

In other words, the C_k, $k\geq 1$, switch on a λ-dependence of the $\rho\to\infty$ asymptotics that is not enforced by the counter terms; cf. eq. (4.19) and the subsequent discussion. Putting them to zero therefore is a natural extension of the minimal subtraction scheme used throughout, whereby one recovers the minimal solution (5.16). In this sense the fixed point (5.16) is unique. The leading quantum correction has the scheme independent coefficient $-2\zeta_2/\zeta_1 = \epsilon$. Finally we cannot resist mentioning the resemblance of (5.16) to the "least coupling" form of the dilaton functional proposed in [40, 41].

For later use let us also prepare an integrated form of (5.15)

$$-\frac{2\pi}{\lambda}\frac{\dot\Xi^3}{\rho} = \widetilde{C}(\lambda) + C^{-1}(\lambda)\frac{1}{h}\,, \tag{5.20}$$

which off-hand gives rise to an integration constant $\widetilde{C}(\lambda)$. However the boundary conditions for $\rho \to \infty$ fix the latter to vanish. This is because with setting discussed in section 4 $h(\rho, \lambda)$ grows at least like ρ^p, $p > 0$. Taking the $\rho \to \infty$ limit of (5.20) then enforces $\widetilde{C}(\lambda) = 0$, as asserted.

Next we consider the linearization of the flow equation (5.6) around the fixed point function $h^{\rm beta}$. In a renormalizable quantum field theory the linearized flow for the essential couplings encodes information about the critical manifold and the rate of approach to it. In our case even the linearized flow equation is an integro-differential equation. Since the lowest order term is fixed by strict renormalizability an appropriate parameterization is

$$\overline{h}(\rho, \lambda) = h^{\rm beta}(\rho, \lambda) + \frac{\lambda}{2\pi}\overline{s}_1(\rho) + \Big(\frac{\lambda}{2\pi}\Big)^2 \overline{s}_2(\rho) + \cdots, \tag{5.21}$$

where the $\overline{s}_l(\rho)$ are functions of ρ and μ which at fixed μ vanish for $\rho \to \infty$. This boundary condition adheres to the "freezing" of the full, non-linear $\overline{h}$-flow at $\rho = \infty$. By (5.18) we also know that μ-independent solutions of the linearized flow equations would have to involve linear combinations of $h_0 \ln^k h_0$, $k \geq 0$, which would again switch on an artificial λ-dependence of the $\rho \to \infty$ asymptotics. We conclude that the properly defined linearized $\overline{h}$-flow does not admit "zero-modes".

We assign to the functions $\overline{s}_l$ a scaling dimension $1 - l$ in order to match the scaling dimensions of the $h_l^{\rm beta}$ under constant rescalings of $h_0 = \rho^p$. Decomposing the linearization of (5.6) into homogeneous pieces results in a recursive system of integro-differential equations for the $\overline{s}_l$, $l \geq 1$. We first anticipate the result and then comment on its derivation and its significance. The result is that for all $l \geq 1$,

$$\overline{s}_l(\rho, \mu) \;\longrightarrow\; 0 \qquad \text{for} \begin{cases} \epsilon = +1 & \text{and} \quad \mu/\mu_0 \to \infty \\ \epsilon = -1 & \text{and} \quad \mu/\mu_0 \to 0\,. \end{cases} \tag{5.22}$$

For $\overline{s}_1$ this arises as follows. The defining equation is

$$2\pi\mu \frac{d}{d\mu}\overline{s}_1 = p\zeta_1 \rho^p \int_\rho^\infty \frac{du}{u^{2p+1}}\overline{s}_1(u) - \frac{\zeta_1}{p}\rho^{1-p}\partial_\rho \overline{s}_1\,. \tag{5.23}$$

It admits a simple generic solution parameterized by a function r_1 of one variable via

$$\overline{s}_1(\rho, \mu) = \rho^p \int_{\rho^p}^\infty \frac{du}{u} r_1\Big(u - \frac{\zeta_1}{2\pi}\ln\frac{\mu}{\mu_0}\Big)\,. \tag{5.24}$$

Here $r_1(u)$ has to decay sufficiently fast for $u \to \infty$ to ensure that the integral converges and vanishes faster than $1/\rho^p$ for large ρ. As long as the flow variable in the argument of (5.24) is positive this translates into decay properties as a function of μ. However since $\zeta_1 = -\epsilon$ the sign $\epsilon = \pm 1$ makes a crucial difference. Under the conditions stated one obtains (5.22). For $\overline{s}_l$, $l > 1$, the derivation of (5.22) employs a recursive solution formula and then proceeds by induction; the proof is somewhat technical and will be described elsewhere.

The behavior (5.22) suggests ultraviolet stability of the fixed point for $\epsilon = +1$ and infrared stability for $\epsilon = -1$. Notably this is *opposite* to the stability pattern of the renormalization flow in O$(1,2)$ models without coupling to gravity, where the $\epsilon = +1$ system is not asymptotically free. Its nonrenormalizable modification though, obtained by truncation of Einstein gravity, is asymptotically safe! In view of the discussion in section 2.2 it is gratifying to see that precisely for the L_+ lagrangian is the coupling flow locally driven toward the fixed point in the ultraviolet. This is because, as argued in section 2.2, *only* for L_+ does the functional integral in (2.11b) plausibly model the truncated 4D quantum gravity in (2.11a). This holds irrespective of the signature of the Killing vectors though for the stationary sector an additional dualization is needed. The existence of an ultraviolet fixed point in this non-renormalizable theory therefore is in the spirit of Weinberg's "asymptotic safety" scenario. To quote from [1]: "A theory is said to be asymptotically safe if the 'essential' coupling parameters approach a fixed-point as the momentum scale of their renormalization point goes to infinity." In a sense elucidated in appendix C the fixed-point can also be regarded as "non-gaussian". (Perhaps it is thus not accidental that 1-loop perturbation theory in the full 4D theory does not seem to display the anti-screening phenomenon [15].) Moreover one would expect that this feature of the truncated theory is a necessary condition for full quantum Einstein gravity to have a non-trivial UV stable fixed point.

On the other hand the present results cannot be subsumed literally into the original asymptotic safety scenario: In the case at hand the space of lagrangians in which the flow moves has no preferred parameterization in terms of (infinitely many) numerical parameters. This is because the bare and renormalized h-functions are related in a nonlinear and nonlocal way (in field space), — a feature one might expect to occur whenever a dimensionless scalar unprotected by symmetries (like a dilaton or the 4D conformal factor) is involved. In particular one cannot classify numerical coupling vectors by their eigenvalues with respect to the gradient-matrix of the beta function and the μ-dependence of the linearized flow will not always be power-like. For example for $n>1$ a choice $r_1(u) \sim e^{-u}u^n$ in (5.24) induces a power-like decay in μ while $r_1(u) \sim u^{-n}$ induces a log-like decay in μ. One might try to classify the functions r_1 by the rate of decay they induce in (5.22) but it is unclear how to 'count' them. In summary, the result (5.22) is in the spirit of the asymptotic safety scenario but there is no obvious way to define the dimension of the critical manifold.

5.3 Flow equations for the fields and "gravitational undressing"

Recall from (4.11) the relation between the bare and the renormalized fields

$$\phi_B^j = \phi^j + \frac{1}{2-d}\Xi^j(\rho,\lambda) + O\left(\frac{1}{(2-d)^2}\right), \tag{5.25}$$

where $\Xi^1 = \Xi^2 = 0$, while Ξ^3, Ξ^4 have been computed in (4.18) and depend on h. Since the bare fields are μ-independent the renormalized fields ϕ^j have to carry an implicit μ-dependence through h. (This is analogous to the situation in an ordinary multiplicatively renormalizable quantum field theory, where the coupling dependence of the wave function renormalization induces a compensating μ-dependence of the renormalized fields governed by the anomalous dimension function.) From (5.25) and the $\overline{h}$-flow (5.6) one derives

$$\mu\frac{d}{d\mu}\overline{\rho} = -\dot{\Xi}^3[\overline{h}](\overline{\rho})\,, \qquad \mu\frac{d}{d\mu}\overline{\sigma} = -\dot{\Xi}^4[\overline{h}](\overline{\rho})\,, \tag{5.26}$$

where $\dot{\Xi}^3[\overline{h}]$, $\dot{\Xi}^4[\overline{h}]$ refer to (5.4) with the solution of (5.6) inserted for h. Note that, conceptually, the problems decouple: One first solves the autonomous equation (5.6) to obtain the coupling flow $\mu \to \overline{h}(\,\cdot\,,\mu)$ which is then used to specify the right hand side of the $\overline{\rho}$-flow equation whose solution in turn determines the $\overline{\sigma}$-flow. For a given solution $\overline{h}$ let $\overline{\phi}^j$ denote the moving field vector and let $\overline{g}_{ij}$ be the target space metric with $\overline{h}$ as a prefactor. The distance-squared traveled along the flow is obtained by integrating

$$\begin{aligned}\overline{g}_{ij}(\overline{\phi})\,\mu\frac{d\overline{\phi}^i}{d\mu}\,\mu\frac{d\overline{\phi}^j}{d\mu} &= \mu\frac{d}{d\mu}\left(\int^{\overline{\rho}}\frac{du}{u}\overline{h}(u,\lambda)\int_u^\infty\frac{dv}{v}\dot{S}[\overline{h}](v,\lambda)\right)\\ &= \mu\frac{d}{d\mu}\Big(-\frac{\lambda}{4\pi}\ln\overline{\rho}^p + O(\lambda^2)\Big)\,.\end{aligned} \tag{5.27}$$

For a given $\overline{h}$-flow it manifestly depends only on the initial and final ρ configuration. In the decoupling limit of constant h the right hand side vanishes identically and the field flow describes null geodesics.

Generally however only the leading terms in $\overline{\rho},\overline{\sigma}$ are readily accessible. Since we insist on having $h(\rho,\lambda) = \rho^p + O(\lambda)$, the leading term on the right hand side of (5.26) is given by $\frac{\lambda}{2\pi}\phi_1^j(\overline{\rho})$, $j = 3,4$, with ϕ_1^j from (4.9). The solution is

$$\begin{aligned}\overline{\rho} &= \rho + \frac{\zeta_1}{p}\frac{\lambda}{2\pi}\rho^{-p+1}\ln\frac{\mu}{\mu_0} + O(\lambda^2)\,,\\ \overline{\sigma} &= \sigma + \frac{\lambda}{2\pi}\left[d_1 - \frac{2a\zeta_1 + p^2}{4bp}\rho^{-p}\right]\ln\frac{\mu}{\mu_0} + O(\lambda^2)\,.\end{aligned} \tag{5.28}$$

In contrast the higher orders are difficult to control.

Although essential for the consistency of the formalism the moving fields $\overline{\rho}(\mu)$ and $\overline{\sigma}(\mu)$ are "inessential" couplings in the sense of [1]. Recall from section 4.2 that the flow of an inessential coupling is effected by field redefinitions and may continue to run even at a fixed point. For $\overline{\rho}(\mu)$ and $\overline{\sigma}(\mu)$ this is almost tautological but it is important for the interpretation of the results: One may observe from (5.28) that the flow pattern of $\overline{\rho}$ is *opposite* to that of the linearized $\overline{h}$-flow in (5.22). For example for $\epsilon = +1$ the value of $\overline{\rho}$ is decreasing with increasing μ/μ_0. Thus if one was to identify $\overline{\rho}$ with an essential coupling its flow would drive it out of the perturbative 'large ρ' regime in the ultraviolet. However $\overline{\rho}$ and $\overline{\sigma}$ cannot be regarded as couplings for at least four reasons: (i) First they meet the defining criterion for being "inessential" discussed in section 4.2. (ii) They continue to run at the fixed point $h^{\rm beta}$. (iii) They are still functions on base space and their value at some point x has no intrinsic meaning. (iv) The x-dependence is such that the currents (3.15), (3.17) with $\overline{\rho},\overline{\sigma}$ inserted fail to be conserved in general.

Properties (i) and (iii) are obvious. Feature (ii) is present already to lowest order in (5.28); a closed expression for the $\overline{\rho}$-flow at the fixed point is given in eq. (5.33) below.

To see (iv) we combine (5.6) and (5.26) to obtain

$$\mu\frac{d}{d\mu}\left(\int_\rho^{\overline{\rho}}\frac{du}{u}\overline{h}(u,\mu)\right) = -\overline{h}(\rho,\mu)\frac{\dot{\Xi}^3[\overline{h}](\rho)}{\rho} = \frac{\lambda}{2\pi}\frac{\zeta_1}{p} + O(\lambda^3)\,, \tag{5.29}$$

where the right hand side is independent of $\overline{\rho}$. As indicated its leading term is also μ-independent so that one recovers (5.28). On the other hand one can decompose the integral as $\int_\rho^\Lambda + \int_\Lambda^{\overline{\rho}}$ with some (x- and μ-independent) constant Λ. The left hand side of (5.29) then splits into two terms the first of which is a harmonic function in x by (5.14), while the right hand side in general is not. For a generic initial h we thus find:

$$\int^{\overline{\rho}}\frac{du}{u}\overline{h}(u,\mu) \text{ is not harmonic}\,. \tag{5.30}$$

This demonstrates (iv) for the current (3.15); a similar analysis could be made for the others. In view of (i)–(iv) we may safely conclude that $\overline{\rho}$ and $\overline{\sigma}$ are "inessential" couplings.

Nevertheless the field flow (5.26) is crucial for the consistency of the formalism. This is highlighted by the pattern that emerges if one inserts $\overline{\rho}$ into the first argument of the running $\overline{h}(\,\cdot\,,\mu)$: Specifically we consider the combination

$$\overline{l}(\mu) := \frac{1}{\overline{h}(\overline{\rho},\mu)} \qquad \text{with } \overline{l}(\mu_0) = \frac{1}{h(\rho)}\,, \tag{5.31}$$

which depends on the value of $h(\rho(x))$ — and hence on x — parametrically through the initial condition. Either by rewriting (5.29) or by direct computation from (5.6) and (5.26) one finds that $\overline{l}$ satisfies (putting $d=2$ for simplicity)

$$\mu\frac{d}{d\mu}\lambda\overline{l} = -\beta_\lambda(\lambda\overline{l})\,. \tag{5.32}$$

This is the usual flow equation for the flat space O(1, 2) sigma-model. The gravitationally dressed functional flow (5.6) has been 'undressed'! This occurs independent of the form of the initial $h(\,\cdot\,,\lambda)$ and may be interpreted as an *equivalence principle* for 2D quantum gravity non-minimally coupled to sigma-models: *By using a scale dependent 'clock field' $\overline{\rho}(x;\mu)$ the effect of 2D quantum gravity on matter can locally be undone.*

Technically this occurs because in defining $\overline{l}(\mu)$ in (5.31) we study the flow of the *numerical value* of $h(\rho)$ with respect to a 'comoving' coordinate system in field space. Since both the $\overline{h}$- and the $\overline{\rho}$-flow were induced by splitting a set of ρ-modified O(1, 2) counter terms (3.23) according to the principle of conformal renormalizability, it is plausible that the relative flow encoded in $\overline{l}(\mu)$ is governed by (5.32). In fact the flow equation (5.32) can also directly be obtained from eq. (4.26), which highlights that the counter terms driving $\overline{l}(\mu)$ are those relating the value of $h_B(\rho_B)$ to the value of $h(\rho)$. Of course in itself (5.32) is of little use, as one is really interested in the flow equation of $\overline{h}(\,\cdot\,,\mu)$ with respect to a *fixed* set of field coordinates. In other words the complexity of the function flows (5.6) and (5.26) has to be addressed because one needs to disentangle the μ-dependence of the function $\overline{h}(\,\cdot\,,\mu)$ from the μ-dependence of its first argument.

JHEP12(2002)066

Ignoring the fact that $\bar{l}$ depends on x, one might be tempted to interpret it as a running coupling. As with $\bar{\rho}$ and $\bar{\sigma}$ this would have the discomforting consequence that the flow $\mu \to \bar{l}(\mu)$ has characteristics opposite to that of the linearized $\bar{h}$-flow (5.24), (5.22). However $\bar{l}$ must again be considered as an *inessential coupling*. One cannot directly apply the variational criterion discussed in section 4.2 because $\bar{l}$ and $\bar{\rho}$ are not independent; so the variation of the lagrangian with respect to $\bar{l}$ would be cumbersome to study. However there are indirect arguments that safely identify $\bar{l}$ as an inessential coupling. Most importantly it continues to flow even after $\bar{h}(\,\cdot\,,\mu)$ reached the fixed point. Since the flow (5.32) is the same for any initial $h(\,\cdot\,)$ one can obtain an explicit formula by evaluating it for $h^{\rm beta}$. This trivializes the $\bar{h}$-flow and the μ-dependence is entirely carried by $\bar{\rho}$. Using (5.20) in (5.29) yields

$$\int_{\rho}^{\bar{\rho}} \frac{du}{u} h^{\rm beta}(u,\lambda) = \frac{\lambda}{2\pi}\frac{1}{C(\lambda)} \ln\frac{\mu}{\mu_0} \qquad \Longleftrightarrow \qquad \mu\frac{d}{d\mu}\ln\bar{\rho}\Big|_{h^{\rm beta}} = \frac{\lambda}{2\pi}\frac{\bar{l}(\mu)}{C(\lambda)}\,, \tag{5.33}$$

where $C(\lambda) = C_0 = p/\zeta_1$ corresponds to the minimal $h^{\rm beta}(\rho)$ in (5.16). Comparing with (5.27) one sees that $\bar{l}(\mu)$ also parameterizes the leading order of the distance squared traveled along the total renormalization flow. $\bar{\rho}|_{h^{\rm beta}}$ now has somewhat nicer properties, but not nice enough: The integral in (5.30) now does define a μ-dependent harmonic function of x. Further, comparing (5.33) with (3.18) one identifies the residual $\bar{\rho}$-flow as geodesic, with $\ln\mu/\mu_0$ playing the role of the affine parameter. However the same is not true for $\bar{\sigma}$, otherwise the integrated distance inferred from (5.27) would vanish. As it doesn't vanish $\bar{l}$ remains an inessential coupling for $h = h^{\rm beta}$.

6. Operator constraints and trace anomaly

Recall from appendix A that the classical hamiltonian and diffeomorphism constraints coincide with the components T_{00} and T_{01}, respectively, of the energy momentum tensor (A.7). The construction of the operator constraints therefore is equivalent to the construction of the renormalized energy momentum tensor $[\![T_{\mu\nu}]\!]$ of the flat space quantum field theory whose (source-extended) renormalized lagrangian was constructed in the previous sections. The so-defined energy momentum operator may be expected to have a non-vanishing trace anomaly $[\![T^{\mu}_{\ \mu}]\!]$. At the fixed point (5.16) of the functional h-flow however one may hope that the anomaly vanishes. In the following we take up these issues consecutively.

6.1 Improved energy momentum tensor

On the bare level the energy momentum operator is uniquely determined by the conservation equation up to an improvement term. It is thus given by (2.8) with constant $a(\rho) = a$, $b(\rho) = b$, modified by the addition of a generic improvement term $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\Phi^B$, whose potential Φ^B replaces f in the last term of (2.8). It is not hard to see that Φ^B can only depend on σ_B and ρ_B and that the counter terms must be σ-independent. The counter terms are in principle determined by the requirement that $T^B_{\mu\nu} = [\![T_{\mu\nu}]\!]$ is a finite composite operator in minimal subtraction whose insertion into correlation functions produces

JHEP12(2002)066

answers for which the UV cutoff can be removed. Combined with the principle of "conformal renormalizability" this turns out to determine the counter terms, and eventually the renormalized improvement potential Φ as the solution of a functional flow equation. For the actual computation of the counter terms it is useful to treat Φ as an arbitrary function of ϕ^j, work out the counter terms and then impose the additional restrictions. In this setting one can again take advantage of the results available in the literature on riemannian sigma-models because improvement terms with a potential Φ correspond to minimal couplings to the scalar curvature of a fiducial background metric in the lagrangian. We refer to appendix B for a compilation of the relevant counter terms.

The restrictive notion of "conformal renormalizability" adopted here for the Ernst-like systems implies that the renormalized improvement potential Φ can only depend on ρ and σ, and that the dependence on σ must be *linear*. To see this we apply the diffeomorphism Ward identity (B.14) to the vector $\partial^i\Phi$ and the action of the Ernst-like systems. This gives

$$[\![\partial^\mu\partial_\mu\Phi]\!] = [\![\nabla_i\nabla_j\Phi\,\partial^\mu\phi^i\partial_\mu\phi^j]\!] - \lambda\frac{\delta S_B}{\delta\phi^j}\partial^j\Phi\,. \tag{6.1}$$

The presence of a $\partial^\mu\sigma\partial_\mu\sigma$ term on the right hand side would destroy one of the conformal Killing vectors on the target space in which case (6.1) would not be a viable improvement for the trace $[\![T^\mu_\mu]\!]$. The absence of such a term requires Φ to be linear in σ. For later convenience we parameterize it as

$$\Phi = f(\rho,\lambda) + f_0(\lambda)\sigma\,, \tag{6.2}$$

where both f and the constant f_0 may depend on λ. The linear σ-dependence has the consequence that also no $\partial^\mu\sigma\partial_\mu\rho$ term appears in the improvement of the trace. Moreover in the counter terms (B.2) relating the bare potential $\Phi_B(\phi_B)$ to the renormalized one $\Phi(\phi)$ the function $f(\rho)$ drops out. To verify this recall that a typical monomial in the differential operator $Z(g)-1$ is of the form $z^{i_1\ldots i_n}(g)\nabla_{i_1}\cdots\nabla_{i_n}$. By an argument similar to the ones in section 3.2 one establishes that only for $n=2$ can this give a non-vanishing contribution upon acting on Φ of the form (6.2). f only appears in the 3-3 component of $\nabla_i\nabla_j\Phi$ and thus disappears upon contraction with a $z^{ij}(g)$ that has vanishing covariant 4-components. We conclude that $[Z(g)-1]\Phi$ is a local function of h whose l-loop contribution scales like Λ^l under $h\to\Lambda^{-1}h$. The contributions coming from $\Psi(g)$ have a similar structure. We write

$$\begin{aligned}
\left[Z\left(\frac{g}{\lambda}\right)-1\right]\Phi + \lambda\Psi\left(\frac{g}{\lambda}\right) &= \frac{1}{2-d}k[h](\rho) + O\left(\frac{1}{(2-d)^2}\right),\\
k(\rho) &= -\frac{f_0}{b}k_1(\rho)\rho\partial_\rho\ln h + hk_2(\rho)\,,\\
k_1(\rho) &= \frac{\lambda}{2\pi}\frac{1}{2h} + O(\lambda^3)\,,\qquad k_2(\rho) = \frac{\lambda}{2\pi}\frac{2}{3h} + \left(\frac{\lambda}{2\pi}\right)^3\frac{1}{12h^3} + O(\lambda^4)\,,
\end{aligned} \tag{6.3}$$

where the explicit form of the low order contributions follows from eqs. (B.6), (B.7). Parameterizing the bare potential as $\Phi_B = f_B(\rho_B) + \mu^{d-2}f_0\,\sigma_B$ one finds from (5.25), (B.2) that f_0 is unrenormalized while the bare function $f_B(\,\cdot\,)$ is related to the renormalized one,

$f(\,\cdot\,)$, by

$$f_B(\rho) = \mu^{d-2} f(\rho) + \frac{\mu^{d-2}}{2-d}[k(\rho) - \partial_\rho f(\rho)\Xi^3(\rho) - f_0 \Xi^4(\rho)] + \cdots . \tag{6.4}$$

In particular the function $f_B(\,\cdot\,)$ will in general not be the same as $f(\,\cdot\,)$. A strict counterpart of the non-renormalization property (4.23) valid for the currents cannot hold therefore. Of course if one takes into account the additional functional change in f the weaker property $T^B_{\mu\nu} = [\![T_{\mu\nu}]\!]$ holds by construction.

The ρ-dependence of f is constrained by its functional flow, which in contrast to that of h is not autonomous. The equation governing the flow $\mu \to \overline{f}(\,\cdot\,,\mu)$ can be obtained either directly from (6.4) or by combining (B.17) with (5.26), keeping $\mu\frac{d}{d\mu} f_0 = (2-d) f_0$ in mind. One finds either way

$$\mu\frac{d}{d\mu}\overline{f} = (2-d)\overline{f} + \partial_\rho \overline{f}\dot{\Xi}^3[\overline{h}] + f_0\dot{\Xi}^4[\overline{h}] - K[\overline{h}]\,, \tag{6.5}$$

where $K[h]$ is obtained from k by substituting $k_1 \to \dot{k}_1$ and $k_2 \to \dot{k}_2$, and the latter can be interpreted either in the sense of (5.3) or (B.16). Note that $K[h] = h\cdot\frac{\delta}{\delta h}k[h] - hk_2[h]$, and $K(\rho) = -\frac{\lambda}{2\pi}\frac{2}{3} + O(1/h)$, for large ρ. For a given $\overline{h}$-flow eq. (6.5) is a linear inhomogeneous equation for $\overline{f}$. We shall only be interested in the solution corresponding to $h^{\rm beta}$, i.e. to the fixed point of the $\overline{h}$-flow. As h is the essential coupling one expects that at its fixed point the form of the improvement potential likewise stabilizes. This consistency condition determines $f^{\rm beta}(\rho) := f[h^{\rm beta}](\rho)$ to be

$$\rho\partial_\rho f^{\rm beta} - f_0\frac{a}{2b} = -\frac{2\pi C}{\lambda} h\Big[K(\rho) + \frac{\lambda}{2\pi}\frac{2}{3}\Big] - \frac{2\pi}{\lambda}\frac{Cf_0}{2b} h \int_\rho^\infty \frac{du}{u}\dot{S}(u,\lambda)\,, \tag{6.6}$$

where we set $d=2$ and the right hand side is evaluated for $h = h^{\rm beta}$. The $\dot{S}$ term enters through eq. (5.4) and we re-adjusted the lower integration boundary so as to extract the constant $-\frac{\lambda}{2\pi}\frac{2}{3f_0}$. The redefined integration boundary was then set to infinity, which removes terms proportional to h in the $\rho \to \infty$ limit. Generally the choice of integration constants affects $f(\rho)$ merely by a shift proportional to the potential of the Noether current (3.15)

$$f(\rho) \;\longrightarrow\; f(\rho) + d(\lambda)\int^\rho \frac{du}{u} h(u,\lambda) \qquad \text{with } d(\lambda) = \sum_{l\geq 1} d_l\Big(\frac{\lambda}{2\pi}\Big)^l . \tag{6.7}$$

Since such a term is already present on the classical level — cf. eq. (2.7) with constant $b(\rho)$ — and one is not forced to modify it, it is natural to impose the absence of ad-hoc λ-dependent corrections to it as a boundary condition.

So far our focus lay on the construction of the renormalized energy momentum tensor. Apart from the ambiguities stemming from the solution of (6.5) it is now fully determined and one can proceed to investigate its trace. The key result to be shown in the next section is

$$\mu\frac{d}{d\mu}\overline{h} = 0 = \mu\frac{d}{d\mu}\overline{f} \iff [\![T^\mu_{\ \mu}]\!] = 0\,. \tag{6.8}$$

That is, the trace anomaly of the improved energy momentum tensor vanishes precisely at the fixed point of the functional flow. Of course h and f are not on the same footing though: h is the essential coupling while f 'merely' defines the proper improvement term.

Technically the equivalence is non-trivial because on both sides different types of information enter. The left hand side contains only information about the basic lagrangian (without sources) and its renormalization. The very definition of $[\![T_{\mu\nu}]\!]$ as a composite operator, on the other hand, requires additional counter terms beyond those needed to renormalize the basic lagrangian. In particular operator mixing takes place, i.e. the counter terms of operators with lower engineering dimension enter. The equivalence (6.8) thus requires both types of counter terms to be correlated. The fact that they indeed are correlated can be traced back to the 'non-renormalization' property of the energy momentum tensor $T^{B}_{\mu\nu} = [\![T_{\mu\nu}]\!]$. The latter gives rise to a precursor of the Curci-Paffuti relation [57] which is instrumental for the proof of (6.8).

6.2 Trace anomaly

We begin with the following expression for the trace anomaly

$$[\![T^{\mu}{}_{\mu}]\!] = \left[\!\!\left[\frac{\lambda}{h}\beta_h\left(\frac{h}{\lambda}\right) L\right]\!\!\right] + \frac{1}{4\pi}[\![(\mathcal{L}_K g_{ij})(\phi)\partial^{\mu}\phi^i\partial_{\mu}\phi^i]\!]\,, \qquad K^j = \frac{2\pi}{\lambda}[W^j - \dot{\Xi}^j + \partial^j\Phi]\,, \quad (6.9)$$

where L is the lagrangian. It is obtained by inserting eq. (5.7) into (B.20) and taking the limit of a flat base space. We already know that the improvement potential Φ is of the form (6.2) with $f(\rho)$ subject to (6.5). The vector W^j is likewise highly constrained. Starting from the definition (B.22) an argument similar to the one yielding (3.19) shows that to all loop orders the covariant vector W_i must be of the form $W_i = (0, 0, W_3(\rho,\lambda), 0)$. Equivalently

$$W^i = \left(0, 0, 0, \frac{\rho}{bh}W_3(\rho,\lambda)\right)\,, \qquad \text{with } W_3(\rho,\lambda) = \left(\frac{\lambda}{2\pi}\right)^3 \frac{1}{8}\partial_{\rho}\left(\frac{1}{h^2}\right) + O(\lambda^4)\,. \qquad (6.10)$$

An important further constraint arises from the 'pre'-Curci-Paffuti relation (B.28). It can be shown to be equivalent to

$$\partial_{\rho}(\dot{\Xi}^3 W_3) = P(\dot{H}) - \partial_{\rho}(h\dot{k}_2) - (\dot{Z}^V)_{3j}\dot{\Xi}^j + \frac{h}{\rho^2}\dot{\Xi}^3\dot{S}\,, \qquad (6.11)$$

with

$$P(\dot{H}) = \dot{N}_3^{\,jk}(\dot{H}(\rho)g_{jk}) - \dot{H}(\rho)\dot{W}_3(\rho)\,.$$

Before proceeding with the general analysis let us briefly check the decoupling limit to the ordinary O(1,2) sigma-model. For a constant $h = 1/\lambda_0$ one expects to recover the trace anomaly of the ordinary O(1,2) sigma-model because the ρ,σ part of the action decouples and is non-interacting. We already saw that for constant h the $\beta_h(h)$ function reduces to the ordinary β_{λ} function, cf. (5.8). When specializing $\dot{\Xi}^3$, $\dot{\Xi}^4$, the integration constants ρ_l in (4.6), (4.7) have to be taken finite, say $\rho_l = \rho_1$, $l \geq 1$. This gives $\dot{\Xi}^3 =$

$\frac{1}{\lambda\lambda_0}\beta_\lambda(\lambda\lambda_0)\rho\ln\rho/\rho_1$, $\dot{\Xi}^4 = -\frac{a}{2b\rho}\dot{\Xi}^3$. When inserted into (6.9) the ρ, σ dependent part in the lagrangian cancels and one ends up with

$$[\![T^\mu{}_\mu]\!]|_{h=1/\lambda_0} = -\frac{1}{2}\frac{1}{(\lambda\lambda_0)^2}\beta_\lambda(\lambda\lambda_0)[\![\partial^\mu n\cdot\partial_\mu n]\!]\,. \tag{6.12}$$

This agrees with the (O(1,2) analogue of the) result in [51] modulo terms proportional to the equations of motion operator. Because of the different schemes used such terms cannot be compared.

An instructive way to proceed with the general analysis is by trying to adjust Φ such that K^j becomes a conformal Killing vector of $g_{ij}(\phi)$,

$$\mathcal{L}_K g_{ij} \stackrel{!}{=} \Omega\, g_{ij}\,. \tag{6.13}$$

Off-hand (6.13) would imply only that the trace anomaly (6.9) is proportional to the lagrangian. In fact it turns out to be equivalent to the vanishing of the anomaly! To show this we parameterize Φ as before, i.e. $\Phi(\rho,\sigma) = f(\rho) + f_0\sigma$. Spelling out the conformal Killing equations for K^j gives rise to a pair of differential equations. After using (5.4) and (6.10) the first one reads

$$\frac{\lambda}{2\pi}\rho\partial_\rho h = C(\lambda)h^2\,\frac{h}{\lambda}\beta_\lambda\Big(\frac{\lambda}{h}\Big) \quad \text{with } C(\lambda) = -\frac{\lambda}{2\pi}\frac{b}{f_0(\lambda)}\,. \tag{6.14}$$

This exactly coincides with eq. (5.15) — which entails the vanishing of the $\beta_h(h)$ function — provided the constants are matched as indicated. Thus the first term on the right hand side of (6.9) vanishes. The second differential equation deriving from (6.13) is

$$h\partial_\rho\left(\frac{\rho\partial_\rho f}{h}\right) = \frac{\lambda}{2\pi}\frac{a}{C}\partial_\rho\ln h + bh\partial_\rho(\dot{\Xi}^4 - W^4)\,, \tag{6.15}$$

and defines $f^{\rm trace}$. Here $\dot{\Xi}^4$ is given in (5.4) and h refers to $h^{\rm beta}$. We postpone the integration of (6.15) for a moment and compute the conformal factor in (6.13)

$$\Omega = \frac{1}{4}g^{ij}\mathcal{L}_K g_{ij} = -\frac{\lambda}{2\pi}\partial_\rho\ln h\Big[\dot{\Xi}^3 + \frac{\lambda}{2\pi}\frac{\rho}{Ch}\Big] = 0\,. \tag{6.16}$$

In the last step the integrated form (5.20) of (6.14) was used. One concludes that K^j is actually a proper Killing vector and the second term in (6.9) vanishes as well.

It remains to integrate equation (6.15). One of the integrations can be performed trivially and gives rise to a λ-dependent integration constant $\widetilde{d}(\lambda)$. Inserting further W^4 from (6.10) and $\dot{\Xi}^4$ from (5.4) some of the terms combine due to (5.20). We absorb $\widetilde{d}(\lambda)$ into the (anyhow unspecified) lower integration boundary of the $\dot{S}$ term. Eventually one ends up with

$$\begin{aligned}\partial_\rho f^{\rm trace} &= -\frac{\lambda}{2\pi}\frac{a}{2C\rho} + \frac{h}{2\rho}\int^\rho \frac{du}{u}\dot{S}(u,\lambda) - W_3(\rho)\,,\\ \Phi^{\rm trace} &= f^{\rm trace}(\rho) - \frac{\lambda}{2\pi}\frac{b}{C}\sigma\,,\end{aligned} \tag{6.17}$$

completing the solution of (6.13). Of course also the vector K^j is determined and comes out to be proportional to $\mathbf{t}_- = (0,0,0,1)$. The proportionality constant parameterizes an additive ambiguity in $\Phi^{\rm trace}$ of the form (6.7). One will naturally set this constant to zero in which case K^j vanishes identically. Equivalently $\dot{\Xi}^j$ is the gradient of a potential

$$\dot{\Xi}^j|_{h^{\rm beta}} = \partial^j(\Phi^{\rm trace} + \omega) \quad \text{with} \quad \omega(\rho,\lambda) = -\int_\rho^\infty du W_3(u,\lambda)\,. \tag{6.18}$$

In summary, we find

$$\mathcal{L}_K g_{ij} \stackrel{!}{=} \Omega\, g_{ij} \implies \beta_h(h) = 0 \qquad \text{and} \qquad K^j = 0 \implies [\![T^\mu{}_\mu]\!] = 0\,. \tag{6.19}$$

Here $\beta_h(h) = 0$ and $K^j = 0$ are equivalent to eqs. (6.14) and (6.17), respectively. We proceed by showing that a converse of the statement (6.19) is also true.

Specifically we verify that the vanishing of the tensorial Weyl anomaly coefficient $B_{ij}(g)$ in eq. (B.21) again implies (6.14) and that the already computed $\Phi^{\rm beta} = f^{\rm beta}(\rho) + f_0\sigma$ is the associated dilaton field. In other words it should come out that

$$B_{ij}(g)\Big|_{h^{\rm beta},\Phi^{\rm trace}} = 0 \qquad \text{and} \qquad \Phi^{\rm trace} = \Phi^{\rm beta}\,. \tag{6.20}$$

In order to verify (6.20) we first compute

$$\sum_{l\geq 1}\Big(\frac{\lambda}{2\pi}\Big)^l l T^{(1,l)}_{ij}(g) = {\rm diag}\left(\frac{\epsilon}{\Delta^2}\frac{h^2}{\lambda}\beta_\lambda\Big(\frac{\lambda}{h}\Big),\ \frac{1}{\Delta^2}\frac{h^2}{\lambda}\beta_\lambda\Big(\frac{\lambda}{h}\Big),\ -\frac{h}{\rho^2}\dot{S}(\rho,\lambda),\ 0\right)\,, \tag{6.21}$$

with $\dot{S}(\rho,\lambda)$ as in (5.4). Further

$$-\frac{2\pi C}{\lambda}\mathcal{L}_{W+\partial\Phi}g_{ij} = {\rm diag}\bigg(\frac{\epsilon}{\Delta^2}\rho\partial_\rho \ln h, \frac{1}{\Delta^2}\rho\partial_\rho\ln h, \frac{a}{\rho}\partial_\rho \ln h - \frac{2\pi C}{\lambda}\frac{2h}{\rho}\partial_\rho\Big[\frac{\rho}{h}(\partial_\rho f + W_3)\Big], 0\bigg). \tag{6.22}$$

with W^j as in (6.10) and Φ of the form $\Phi(\phi,\lambda) = f(\rho,\lambda) - \frac{\lambda}{2\pi}\frac{b}{C}\sigma$, as before. Together the condition $B_{ij} = 0$ translates into just two differential equations. They can be seen to coincide with (6.14) and (6.17), as asserted. Thus:

$$[\![T^\mu{}_\mu]\!] \stackrel{!}{=} 0 \quad \implies \quad \beta_h(h) = 0 \quad \text{and} \quad K^j = 0\,. \tag{6.23}$$

In particular the differential equations (6.14) and (6.17) are necessary and sufficient conditions for the vanishing of the trace anomaly. The first one determines $h = h^{\rm beta}$ and the other one the improvement potential $\Phi^{\rm trace}$. An instructive consistency check is obtained by starting from the alternative expression (B.27) for $[\![T^\mu_\mu]\!]$: Using the non-renormalization of the conserved current $h(\rho)\partial_\mu \ln\rho$ — cf. eq. (4.23) — the total divergence term $\partial^\mu[\![\partial_\mu \rho W_3(\rho)]\!]$ can be rewritten as $[\![\rho^{-1}\partial^\mu\rho\partial_\mu\rho\, h\partial_\rho(\rho W_3/h)]\!]$. Inserting further (6.1) and (6.21) the vanishing of $[\![T^\mu_\mu]\!]$ translates into the previously found differential equations.

Before proceeding let us briefly note the corresponding results in the abelian subsector. We already know from (5.11), (5.12) that $h(\cdot)$ and ρ remain unrenormalized, while σ is renormalized (only) at the 1-loop level. Using this, one finds that the only way to solve $B_{ij} = 0$ (with a ρ-dependent h) is by having $\Phi(\phi, \lambda) = f(\rho, \lambda)$ independent of σ. Then $h(\rho, \lambda)$ turns out to be unconstrained and only the counterpart of the differential equation (6.17) for $f(\rho, \lambda) = \sum_{l \geq 1} (\frac{\lambda}{2\pi})^l f_l(\rho)$ has to be solved. Taking $h(\rho) = \rho^p$, the 1-loop contribution is $f_1(\rho) = \frac{p}{8} \ln \rho$. Just as with Ξ^4 we expect all higher contributions to vanish, rendering also the distinction between $f^{\rm beta}$ and $f^{\rm trace}$ superfluous.

In the nonabelian system $f^{\rm beta}$ and $f^{\rm trace}$ are defined through very different conditions. Their equivalence, as asserted in the second part of (6.20) is also needed to conclude the derivation of (6.8). We now show that indeed

$$f^{\rm trace} \simeq f^{\rm beta} , \tag{6.24}$$

where '$\simeq$' denotes equality modulo (6.7). Matching the expressions in (6.17) and (6.6) one finds that (6.24) requires the following identity

$$\rho W_3(\rho) = \frac{2\pi C}{\lambda} h\Big(K[h] + \frac{\lambda}{2\pi}\frac{2}{3}\Big) - h\int_\rho^\infty \frac{du}{u}\dot{S}(u, \lambda) \qquad \text{for } h = h^{\rm beta} . \tag{6.25}$$

Luckily this indeed is an identity; it arises from the 'pre'-Curci-Paffuti relation (6.11) as follows: from eq. (B.12) we have $(\dot{Z}^V)_i^{\,j} \partial_j V = \partial_i(\dot{Z} V)$ for any scalar V. Applied to $\omega + \Phi$ in (6.18) one finds $(\dot{Z}^V)_3^{\,j}\, \dot{\Xi}^j = -\frac{f_0}{b}\partial_\rho[\dot{k}_1(\rho)\rho\partial_\rho \ln h]$. Keeping eq. (5.20) and the definition of K in (6.5) in mind one arrives at (6.25). As an explicit check one can verify that the right hand side vanishes up to and including $O(\lambda^2)$, consistent with $W_3(\rho) = O(\lambda^3)$.

In summary the improvement potential stationary at the fixed point coincides — to all loop orders — with the one that cancels the trace anomaly. The improved energy momentum tensor differs from that in (A.7) by an improvement term

$$\Delta T_{\mu\nu} = (\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^\kappa \partial_\kappa)\, \Phi^{\rm trace} , \tag{6.26}$$

with $\Phi^{\rm trace} = f(\rho) - \frac{\zeta_1 b}{p}\sigma = \Phi^{\rm beta}$ determined by (6.17) or (6.6). It is separately conserved and cancels the unwanted quantum corrections to the trace of (A.7).

A well-known consequence of the Curci-Paffuti relation is that the Weyl anomaly coefficient B^Φ is a constant whenever B_{ij} vanishes. In view of (6.20) this should now come out automatically. B^Φ can be parameterized in terms of $K[h]$ and the improvement potential (6.17) as

$$\lambda B^\Phi \left(\Phi, \frac{g}{\lambda}\right) = -K[h] + \frac{f_0}{bh}\Big[2\rho\partial_\rho f - \frac{a}{b} f_0 + \rho W_3\Big] . \tag{6.27}$$

Combining (6.25) with (6.24) one finds indeed

$$\lambda B^\Phi \left(\Phi, \frac{g}{\lambda}\right)\Big|_{h^{\rm beta}, \Phi^{\rm trace}} = \frac{\lambda}{2\pi}\frac{2}{3} , \quad \text{i.e.} \quad c = 4 . \tag{6.28}$$

Here $c = 4$ is the formal central charge of the improved energy momentum tensor at the fixed point. Note that we established (6.28) to all loop orders despite the fact that the explicit form of (6.27) is known only at low orders. The specific value 2/3 for the constant hinges on a 'natural' choice for the various integration constants involved. This is best illustrated by an explicit computation.

Since B^Φ is known explicitly up to and including $O(\lambda^2)$ one can solve the differential equations (6.14), (6.17) to the same order and verify that B^Φ comes out to be a constant to that order upon inserting the general solutions. This is what we shall do now. One starts by expanding the right hand side of (6.17) in powers of λ, inserts the general solutions for $h_0^{\rm beta}$, $h_1^{\rm beta}$ from (5.18) and performs the ρ-integration. Only $C_0 = -\epsilon p$ and an arbitrary constant C_1 enter. In order to illustrate its impact we also modify $h_1^{\rm beta}(\rho)$ by a trivial additive contribution $t_1\rho^p$, i.e. we use

$$h^{\rm beta}(\rho) = \rho^p + \frac{\lambda}{2\pi}\epsilon\Big(1 - C_1\rho^p\ln\rho + t_1\rho^p\Big) + O(\lambda^2)\,. \tag{6.29}$$

After some computation one finds the following general solution depending on the power p and the integration constants C_1, d_1, d_2, while t_1 drops out:

$$\Phi^{\rm trace}(\phi) = \frac{\lambda}{2\pi}\Phi_1(\rho,\sigma) + \Big(\frac{\lambda}{2\pi}\Big)^2\Phi_2(\rho,\sigma) + O(\lambda^3)\,, \tag{6.30a}$$

$$\Phi_1(\rho,\sigma) = \frac{1}{4p}[2\epsilon a + p^2]\,\ln\rho + d_1\,\rho^p + \frac{\epsilon b}{p}\,\sigma\,, \tag{6.30b}$$

$$\Phi_2(\rho,\sigma) = \frac{3\epsilon}{8}\rho^{-p} + \left[\frac{C_1}{4}\Big(\frac{2a}{p^2} - \epsilon\Big) + \epsilon d_1 p\right]\ln\rho + \rho^p[d_2 - \epsilon C_1 d_1\ln\rho] + \frac{bC_1}{p^2}\sigma\,. \tag{6.30c}$$

Both in Φ_1 and Φ_2 an irrelevant additive constant has been omitted. Evaluated on this solution B^Φ is field independent as it should and comes out as

$$\lambda B^\Phi\Big(\Phi, \frac{g}{\lambda}\Big)\bigg|_{h^{\rm beta},\Phi^{\rm trace}} = \frac{\lambda}{2\pi}\frac{2}{3} + \Big(\frac{\lambda}{2\pi}\Big)^2 2\epsilon d_1 + \Big(\frac{\lambda}{2\pi}\Big)^3 2(\epsilon d_2 - d_1 t_1) + O(\lambda^4)\,. \tag{6.31}$$

Note that the result is independent of the parameters $p = \zeta_1 C_0$ and C_1 entering $h^{\rm beta}$. The parameter t_1 introduced for illustration merely changes the overall normalization of $h_0(\rho)$, and hence of the tree-level lagrangian. It should clearly be set to zero. The constants d_1, d_2, etc enter through the integration of (6.17) and modify the large ρ asymptotics of $\rho\partial_\rho\Phi$. If they are put to zero $\frac{2\pi}{\lambda}\rho\partial_\rho\Phi$ approaches the constant $-a/(2C(\lambda)) + p/4$ for $\rho\to\infty$, while switching them on produces a power-like asymptotics of the form ρ^p or $\rho^{p-1}\ln\rho$, etc. Again, this is not enforced by the counter terms and one would naturally stipulate that the large ρ asymptotics of $\frac{2\pi}{\lambda}\rho\partial_\rho\Phi$ is given by the λ-independent 1-loop constant $\epsilon a/2p + p/4$. This is the same as removing additive contributions of the form (6.7). Doing so one recovers the 'canonical' value $\lambda B^\Phi(\Phi, g/\lambda) = \frac{2}{3}\frac{\lambda}{2\pi}$, i.e. $c = 4$.

Based on the results of [60] one then expects that for the energy momentum tensor improved via (6.26) the combinations $[\![T_{\pm\pm}]\!] \sim [\![\mathcal{H}_0 \pm \mathcal{H}_1]\!]$ generate a 2D conformal algebra with formal central charge $c = 4$. The value $c = 4$ in itself has little significance because the state space generated by $\rho, \partial_\mu\sigma$ and e.g. the Noether currents has indefinite metric. The latter feature is not an artifact, it is directly related to the notorious "conformal factor problem" of 4D quantum Einstein gravity. As stressed in [67], even for free field doublets of opposite signature inequivalent quantizations exist which affect the value of c through the choice of vacuum. In the case at hand quantum counterparts of the non-local charges mentioned after eq. (1.1) together with the quantities found in [28] are candidates for quantum observables which collectively should generate the physical state space. A complete construction of these observables in a lagrangian-based formulation is likely to be very difficult. As with other integrable field theories, however, one can try to accumulate evidence that both of these vastly different formulations actually describe the same system. For the 2-Killing vector reduction already the successful construction of one nonlocal observable would presumably entail the factorization properties characteristic for 'integrability', — and hence would strongly indicate that the non-perturbative bootstrap formulation of [28] and the present lagrangian-based quantum theory coincide.

7. Conclusions

The main result obtained here is that truncations of quantum Einstein gravity corresponding to geometries with two Killing vectors are asymptotically safe. The truncated systems can be renormalized at the expense of introducing infinitely many essential couplings that are combined into the function $h(\,\cdot\,)$. The renormalization flow then has a unique ultraviolet stable fixed point at which the trace anomaly vanishes. This holds irrespective of the signature of the Killing vectors, although in the stationary sector a dualization is needed in order to see this. The significance of the result for the full 4D theory is that the asymptotic safety scenario passes an important self-consistency test: if it is true for the full theory it should be true in every truncation that preserves the presumed 'ferromagnetic' nature of the selfcoupling (but not vice versa). Heuristically one would expect that simply 'freezing' fluctuations transversal to the Killing orbits meets this criterion.

We may also offer some comments on future directions. For pure Einstein gravity as considered here an important open problem is to link the present lagrangian-based Dirac quantization to the bootstrap formulation of [28]. Concretely this can be done by studying the conditions under which the first non-local charge survives quantization. Following Lüscher's strategy in the O(3) nonlinear sigma-model (without coupling to gravity) [50] one will check for the absence of unwanted dimension two operators in the operator product expansion of two Noether currents, which ideally would be the case at the fixed point. An early example for such a match between integrability and the vanishing of a beta function was found in [49]. Understanding the relation between Dirac- and reduced phase space quantization is also required to identify the origin of the spontaneous $\mathrm{O}(1,2)$ symmetry-breaking found in [28]: is it induced by the projection onto the physical state space or already present on the enlarged state space providing the 'arena' for the constraints?

Another strand to be taken up is the inclusion of matter. Classical integrability is known to be preserved in the 2-Killing vector reduction of a wide class of matter extensions [71]. They range from Einstein-Maxwell over dilaton-axion gravity [69, 70] to $N=16$ supergravity [68]. The renormalizability and fixed point structure of all these systems can be investigated by the techniques developed here. In particular a potential relation to [40, 41] should be worth exploring.

Finally, since already one Killing vector is enough to produce the coset structure [71] which has been instrumental here, one might hope that similar ideas apply in these yet larger sectors, eventually helping to 'tame' gravity's non-renormalizability.

JHEP12(2002)066

Acknowledgments

I wish to thank P. Forgács for the enjoyable collaboration in part of this project and important discussions throughout. In addition I profited from conversations and correspondence with J. Balog, T. Jacobson, D. Maison, N. Mohammedi, M. Reuter, C. Torre, and P. Weisz. Part of this work was done at the Max-Planck-Institut für Physik, whose hospitality is gratefully acknowledged.

A. The 2-Killing vector reduction of general relativity

The solutions of Einstein's equations with two Killing vectors cover a variety of physically interesting situations: If one of the Killing vectors is timelike these are stationary axisymmetric spacetimes, among them in particular all the prominent black hole solutions. If both Killing vectors are spacelike the subsector comprises (depending on the sign of $\partial^\mu\rho\partial_\mu\rho$ with ρ defined below) cylindrical gravitational waves, colliding plane gravitational waves, as well as generalized Gowdy cosmologies. In contrast to the spherical reduction and the matter-coupled systems based on it, here one is dealing with infinitely many nonlinearly self-interacting gravitational degrees of freedom. In a hamiltonian formulation this results (with and without matter) in a 'kinematical' diffeomorphism and a 'dynamical' hamiltonian constraint, very much like in the full theory. Unlike in the full theory, however, an infinite set of (nonlocal) observables Poisson commuting with all the constraints can be constructed explicitly; see [28, 29, 31] and the references therein.

In order to fix notations and conventions we recall here the main steps of the reduction procedure; see also [21] for a recent review. Our spacetime conventions are that of Landau-Lifshitz, The classical theory of fields, editions after 1971. In the classification of Misner-Thorne-Wheeler these are $(-,+,+)$ conventions for the metric, Riemann tensor, Einstein tensor, respectively. In particular the spacetime metric has signature $(+,-,-,-)$, and is denoted by G_{MN} (in abstract index notation). As usual it is convenient to adopt a coordinate system, say (x^0,x^1,y^1,y^2), in which the Killing vector fields act as coordinate derivatives, $\frac{\partial}{\partial y^1}$, $\frac{\partial}{\partial y^2}$. An ansatz for the 4D line element in these coordinates will then be parameterized by functions depending on $x=(x^0,x^1)$ only. Further it is convenient to treat both possible signatures of the Killing vectors simultaneously. We distinguish both cases by a sign, such that $\epsilon=+1$ corresponds to both Killing vectors being spacelike and $\epsilon=-1$ corresponds to one being spacelike and the other being timelike. For the reduction one then has two options: (a) direct reduction, and (b) reduction and dualization. For $\epsilon=+1$ both procedures lead to identical actions. For $\epsilon=-1$ the reduced actions are classically equivalent but differ by a crucial signature which (most likely) leads to distinct quantum theories.

(a) Direct reduction: the ansatz for the line element reads [19, 18]

$$\epsilon dS^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu - \rho(x)M_{ab}(x)dy^a dy^b\,. \tag{A.1}$$

Here $\gamma_{\mu\nu}(x)$ is a 2D metric with lorentzian signature if $\epsilon=+1$ and with euclidean signature if $\epsilon=-1$. $M_{ab}(x)$ is a symmetric 2×2 matrix normalized to have determinant $\det M=\epsilon$

(so that $\det G < 0$ always). In particular with these conventions we can assume $\rho \geq 0$ for the degree of freedom parameterizing the determinant. In the axisymmetric case y^2 is the time variable and the overall $\epsilon = -1$ sign on the right hand side of (A.1) is needed to restore the $(1, -1, -1, -1)$ signature. In order to minimize the number of sign flips in the 2D theory we shall base the block decomposition on the 4D metric ϵG_{MN}. Since the 4D Ricci scalar changes sign under a sign flip of the metric this can be compensated simply by multiplying the reduced action by ϵ.

We shall assume throughout that the metric $\gamma_{\mu\nu}(x)$ is conformally flat, i.e. that by a diffeomorphism in the (x^0, x^1) coordinates it can be brought into the form

$$\gamma_{\mu\nu} = \eta_{\mu\nu}\, e^{\sigma}\,, \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix} \qquad \text{with } \epsilon = \pm 1\,. \tag{A.2}$$

For $\epsilon = 1$ this is no restriction, for $\epsilon = -1$ there could be topological obstructions to achieving (A.2) globally. For the matrix M often a parameterization in terms of 'hyperbolic spins' n^j, $j = 0, 1, 2$, is useful, where $(n^0)^2 - (n^1)^2 - (n^2)^2 =: n \cdot n = \epsilon$. Explicitly

$$M = \begin{pmatrix} n^0 + n^2 & -n^1\,, \\ -n^1 & n^0 - n^2 \end{pmatrix}, \qquad \det M = \epsilon\,, \qquad n^j \tau_j = M\, \tau_0\,, \tag{A.3}$$

where τ_j, $j = 0, 1, 2$, is a basis of $sl(2, \mathbb{R})$. Frame rotations in the Killing coordinates then induce $\mathrm{O}(1,2)$ rotations of the n^j, via

$$\begin{pmatrix} dy^1 \\ dy^2 \end{pmatrix} \longrightarrow A^T \begin{pmatrix} dy^1 \\ dy^2 \end{pmatrix}, \quad A \in \mathrm{SL}(2, \mathbb{R}) \quad \Longrightarrow \quad n^j \tau_j \longrightarrow A (n^j \tau_j) A^{-1} = (n^j \Lambda_j^{\ k}) \tau_k\,,$$

with an $\mathrm{O}(1,2)$ matrix Λ. By definition $\mathrm{O}(1,2)$ preserves the constraint $n \cdot n = \epsilon$ and thus is the symmetry group of a 2D hyperboloid H_ϵ. The sign ϵ determines whether the hyperboloid is one- or two-sheeted; in the latter case we restrict attention to one branch (leaving only $\mathrm{SO}(1,2)$ as the invariance group). Explicitly

$$\begin{aligned} H_+ &= \{ n \in \mathbb{R}^{1,2} \,|\, n \cdot n = 1,\, n^0 > 0 \} \quad \text{two-sheeted hyperboloid}\,, \\ H_- &= \{ n \in \mathbb{R}^{1,2} \,|\, n \cdot n = -1 \} \quad \text{one-sheeted hyperboloid}\,. \end{aligned} \tag{A.4}$$

Both hyperboloids are (pseudo-) riemannian spaces of constant negative curvature, normalized to -2 in our conventions. We write $ds^2_{H_\epsilon} = (d\Delta^2 + \epsilon dB^2)/\Delta^2$ for the metric in canonical coordinates (Δ, B). The latter are related to the hyperbolic spins by

$$n^0 = \frac{1 + \epsilon\Delta^2 + B^2}{2\Delta}\,, \qquad n^1 = -\frac{B}{\Delta}\,, \qquad n^2 = \frac{-1 + \epsilon\Delta^2 + B^2}{2\Delta}\,. \tag{A.5}$$

For the matrix M it amounts to a Gauss decomposition

$$M = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon\Delta & 0 \\ 0 & \Delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

and the line element (A.1) takes the form

$$dS^2 = e^{\sigma}[\epsilon (dx^0)^2 - (dx^1)^2] - \epsilon \frac{\rho}{\Delta} (dy^2 + B dy^1)^2 - \rho\Delta (dy^1)^2\,. \tag{A.6}$$

The complex combination $E = \Delta + iB$ is the Ernst potential [19]. The signature of the Killing vectors poses the constraints $\rho/\Delta > 0$ and $\Delta^2 + \epsilon B^2 > 0$.

Inserting the ansatz (A.6) into the Einstein equations a system of partial differential equations is obtained which on general grounds (see [22] for a survey) coincide with the Euler-Lagrange equations following from the reduced action (2.2). Importantly also the symplectic structure following from the 2D action coincides with the restriction of the symplectic structure one has on the full phase space of general relativity. This is crucial with regard to quantization. Strictly speaking this equivalence of the symplectic structures has been shown only for the $\epsilon = 1$ case, but most likely it also holds in the axisymmetric sector; see e.g. [23, 22] for a discussion. A complete description of the phase space in addition requires the specification of boundary or fall-off conditions for the fields. For the cylindrical waves and the generalized Gowdy cosmologies this is available in the literature [23]; for the other sectors the results are incomplete. For the development of a perturbative quantum theory, however, not all of the subtle differences in signatures and boundary terms are important. The essential dynamical information about the phase space (as embedded into the full phase space of general relativity) is contained in the 2D reduced action. In the bulk of the article we therefore distinguish only the two main situations — one or both Killing vectors spacelike — by a sign ($\epsilon = -1, 1$, respectively) and try to develop the framework for all subsectors simultaneously, starting directly from the reduced action.

The constraints associated with the 2D diffeomorphism invariance of (2.2) are obtained by the familiar ADM prodecure, see e.g. [23] in the present context. A technically convenient shortcut is to compute the "would-be" energy momentum tensor by varying the action $S_\epsilon = \int d^2x L_\epsilon$ with respect to the metric. One finds

$$\begin{aligned} \lambda\, T_{\mu\nu} = \frac{\lambda}{\sqrt{\gamma}} \frac{\delta S_\epsilon}{\delta\gamma^{\mu\nu}}\bigg|_{\gamma = e^\sigma \eta} &= -\rho\Big(\partial_\mu n\cdot\partial_\nu n - \eta_{\mu\nu}\frac{1}{2}\partial^\kappa n\cdot\partial_\kappa n\Big) - \\ &\quad - \epsilon\Big(\partial_\mu\rho\partial_\nu + \partial_\nu\rho\partial_\mu - \eta_{\mu\nu}\,\partial^\kappa\rho\partial_\kappa\Big)\Big(\sigma + \frac{1}{2}\ln\rho\Big) + \\ &\quad + 2\epsilon(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^\kappa\partial_\kappa)\rho\,. \end{aligned} \tag{A.7}$$

Re-expressed in terms of the momenta T_{00} coincides with the hamiltonian constraint $\mathcal{H}_0$ and T_{01} coincides with the 1D diffeomorphism constraint $\mathcal{H}_1$. Due to the last ('anti-improvement') term in (A.7) the trace is non-zero, $\lambda\, T^\mu{}_\mu = -2\epsilon\partial^\mu\partial_\mu\rho$. Its form however complies with conformal invariance of the flat space system; see e.g. [64]. More directly the trace can also be obtained by varying with respect to σ:

$$T^\mu{}_\mu = -2\frac{\delta}{\delta\sigma}\int d^2x\, L_\epsilon(n,\rho,\sigma)\,.$$

The overall sign in (A.7) has been chosen such that upon reduction ρ, σ =const, $T_{\mu\nu}$ becomes the energy-momentum tensor of the O$(1,2)$ nonlinear sigma-model. Its energy density then is positive semi-definite for $n\cdot n = 1$ and indefinite for $n\cdot n = -1$. Among the 2D Weyl transformations in eq. (2.5) with $\nabla^2\omega = 0$ are those induced by diffeomorphisms of the form $x^\pm \to f^\pm(x^\pm)$, where $x^\pm$ are lightcone coordinates with respect to the flat metric $\eta_{\mu\nu}$. Our conventions are $x^\pm = (x^0 \pm x^1)/2$ for $\epsilon = 1$ and $x^\pm = (x^0 \pm ix^1)/2$ for $\epsilon = -1$. We write $\partial_\pm = \partial_0 \pm \sqrt{\epsilon}\partial_1$, with $\sqrt{\epsilon} = 1, -i$, for $\epsilon = 1, -1$, respectively. The transformations

JHEP12(2002)066

$x^\pm \to f^\pm(x^\pm)$ are the usual 2D conformal transformations that preserve the conformally flat form of the metric with $\sigma \to \sigma - \ln \partial_+ f^+ \partial_- f^-$. The lightcone components $T_{\pm\pm}$ transform covariantly as second rank tensors since the non-covariance of σ cancels that of $\partial_+^2 \rho$. The trace T_{+-}, non-zero off-shell, could be canceled by switching to an improved $T_{\mu\nu}$ in the usual way. Technically it is often simpler to put ρ on shell, $\partial_+\partial_-\rho = 0$, and express $T_{\pm\pm}$ in terms of the conformal scalar $\sigma - \frac{1}{2}\ln(\partial_+\rho\partial_-\rho)^2$ [27].

In conformal gauge $\gamma_{\mu\nu} = e^\sigma \eta_{\mu\nu}$ the action based on (2.2) becomes that of a flat space sigma-model

$$S_\epsilon = -\frac{1}{2\lambda}\int d^2x \, [\rho \partial^\mu n \cdot \partial_\mu n + \epsilon \partial^\mu \rho \partial_\mu (2\sigma + \ln \rho)] \,. \tag{A.8}$$

The $T_{\pm\pm}$ constitute its energy-momentum tensor and the gravitational origin of the system enters only through the vanishing conditions $T_{\pm\pm} \approx 0$. They can be verified to be first class constraints and to generate two commuting copies of a (centerless) Virasoro-Witt algebra with respect to the Poisson structure induced by (A.8). On general grounds the Einstein equations for the metrics (A.1) will coincide with those obtained by variation of the reduced action (A.8). The equations of motion are

$$\begin{aligned}
&\partial_\mu \partial^\mu \rho = 0 \,, \qquad \partial^\mu \partial_\mu (2\sigma + \ln \rho) = \epsilon \partial^\mu n \cdot \partial_\mu n \,, \\
&\partial^\mu (\rho \partial_\mu n^j) + \epsilon \rho (\partial^\mu n \cdot \partial_\mu n) n^j = 0 \,.
\end{aligned} \tag{A.9}$$

In particular they ensure the consistency conditions $\partial_- T_{++} = 0 = \partial_+ T_{--}$.

(*b*) **Dual action:** recall that in canonical coordinates (Δ, B) one has $\partial^\mu n \cdot \partial_\mu n = -(\epsilon \partial^\mu \Delta \partial_\mu \Delta + \partial^\mu B \partial_\mu B)/\Delta^2$. The action (A.8) thus has a manifest $B \to B + \text{const}$ invariance. It is instructive to perform an abelian T-duality transformation (in the sense of Buscher [65]) with respect to this symmetry. Denoting the dual field by $B_{\scriptscriptstyle D}$ the resulting lagrangian is

$$\begin{aligned}
L_\epsilon^{\scriptscriptstyle D} = &\frac{1}{2\lambda}\Big[\epsilon \frac{\rho}{\Delta^2} \partial^\mu \Delta \partial_\mu \Delta + \epsilon \frac{\Delta^2}{\rho} \partial^\mu B_{\scriptscriptstyle D} \partial_\mu B_{\scriptscriptstyle D} - \epsilon \rho^{-1} \partial^\mu \rho \partial_\mu \rho - 2\epsilon \rho^{-1} \partial^\mu \rho \partial_\mu \sigma\Big] + \\
&+ \epsilon \partial^\mu \left[\rho \partial_\mu \left(\sigma + \ln\left(\frac{\rho^{1/2}}{\Delta}\right)\right) + 2\partial_\mu \rho\right] .
\end{aligned} \tag{A.10}$$

The total derivatives are introduced for later use. By means of an involutive field redefinition the dual lagrangian can *almost* be brought back into the original form

$$\begin{aligned}
&L_\epsilon^{\scriptscriptstyle D}(\phi_{\scriptscriptstyle D})) = \epsilon L_+(\eta(\phi_{\scriptscriptstyle D})) \,, \qquad \eta^2 = \text{id} \,, \\
&\eta(\Delta, B_{\scriptscriptstyle D}, \rho, \sigma) = \Big(\frac{\rho}{\Delta}, B_{\scriptscriptstyle D}, \rho, \sigma + \frac{1}{2}\ln \rho - \ln \Delta\Big) =: (\Delta_{\scriptscriptstyle D}, B_{\scriptscriptstyle D}, \rho, \sigma_{\scriptscriptstyle D}) \,.
\end{aligned} \tag{A.11}$$

Here $\phi_{\scriptscriptstyle D} = (\Delta, B_{\scriptscriptstyle D}, \rho, \sigma)$ and L_+ is the lagrangian (2.2) in conformal gauge ($\gamma_{\mu\nu} = e^\sigma \eta_{\mu\nu}$), including total derivatives. In our conventions $\partial_\mu B_{\scriptscriptstyle D} = -\epsilon \rho \Delta^{-2} \epsilon_{\mu\nu} \partial^\nu B$, on shell. One notices an important (convention-independent) difference between the situation when both Killing vectors are spacelike ($\epsilon = 1$) and when one is timelike ($\epsilon = -1$). For $\epsilon = +1$ the signature of the target space remains unchanged under dualization while for $\epsilon = -1$ it changes into $(-,-,+,-)$.[5] As a consequence the action (A.10) can only for $\epsilon = +1$ be obtained directly from a local and real 4D line element, namely

$$dS_D^2 = \frac{\rho^{1/2}}{\Delta} e^{\sigma}[\epsilon(dx^0)^2 - (dx^1)^2] - \epsilon\Delta\,(dy^2 + \psi dy^1)^2 - \frac{\rho^2}{\Delta}(dy^1)^2\,, \tag{A.12}$$

with $\epsilon = +1$. For $\epsilon = -1$ the line element (A.12) describes stationary axisymmetric solutions; the spacelike/timelike nature of the two Killing vectors $\partial/\partial y^1$ and $\partial/\partial y^2$ poses the conditions $\Delta > 0$ and $\rho^2 + \epsilon\psi^2\Delta^2 > 0$. Evaluating the Einstein-Hilbert lagrangian on (A.12) gives

$$\begin{aligned} -\sqrt{-\det G_D}\, R(G_D) \;=\; & \frac{1}{2}\left[\epsilon\frac{\rho}{\Delta^2}\partial^\mu\Delta\partial_\mu\Delta + \frac{\Delta^2}{\rho}\partial^\mu\psi\partial_\mu\psi - \epsilon\frac{\partial^\mu\rho\partial_\mu\rho}{\rho} - 2\epsilon\partial^\mu\rho\partial_\mu\sigma\right] + \\ & + \epsilon\partial^\mu\left[\rho\partial_\mu\left(\sigma + \ln\left(\frac{\rho^{1/2}}{\Delta}\right)\right) + 2\partial_\mu\rho\right]. \end{aligned} \tag{A.13}$$

One sees that for $\epsilon = 1$ the identification $\psi = B_D$ reproduces (A.10). In contrast for $\epsilon = -1$ a purely imaginary dual field would be needed $\psi = \pm iB_D$ to match (A.13) and (A.10), which would however make the line element (A.12) complex. The mapping $(\Delta, B_D, \rho, \sigma) \to (\Delta_D, \sqrt{\epsilon}B, \rho, \sigma_D)$ which in modern terminology brings the T-dual action back into identically the original form (with fields local with respect to the line element) is known as the Kramer-Neugebauer involution [20]. Its existence is closely related to the classical integrability of the system. Because of the signature change no local real line element can reproduce (A.10) in the Einstein-Hilbert lagrangian. Nevertheless also for $\epsilon = -1$ is the dual lagrangian classically equivalent to the original one. For the equations of motion this holds by construction, for the Poisson structure one can check directly that the transition from L_ϵ in (2.2) to L_ϵ^D in (A.10) is a canonical transformation.

In summary, after dualization both sectors $\epsilon = 1$ and $\epsilon = -1$ are described by reduced actions whose 'matter' part corresponds to a sigma-model on a noncompact riemannian (rather than pseudo-riemannian) symmetric space, i.e. the hyperboloid H_+ in (A.4).

B. Renormalization of riemannian sigma-models

For convenient reference we review here some aspects of the renormalization of riemannian sigma-models. We largely follow the thorough treatment of Osborn [59]. As usual we adopt the covariant background field expansion, dimensional regularization and minimal subtraction. For the purposes of renormalization it is useful to consider an extended lagrangian of the form

$$\lambda L(G;\phi) = \frac{1}{2}\widehat{\gamma}^{\mu\nu} g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \widehat{\gamma}^{\mu\nu}\partial_\mu\phi^i V_{\nu i} + \frac{1}{2}R^{(2)}(\widehat{\gamma})\Phi + F\,. \tag{B.1}$$

[5] The same result is obtained by first performing the reduction from 4 to 3 dimensions, dualizing the 3D theory and then reducing with respect to the second Killing vector [71, 21]. The additional sign flip as compared to the usual dualization formulas can also be understood in terms of the familar (formal) functional integral argument; cf. section 2.2.

Here $G = \{g, V, \Phi, F\}$, $G = G(\phi; x)$ is a collection of generalized couplings/sources (of the tensor type indicated by the index structure) that depend both on the fields ϕ^j and explicitly on the point x in the "base space". The latter is a 2-dimensional riemannian space with metric $\widehat{\gamma}_{\mu\nu}(x)$, extended to d dimensions in the sense of dimensional regularization, and $R^{(d)}(\widehat{\gamma}) = R^{(2)}(\widehat{\gamma})/(d-1)$. The action functional is $S[G;\phi] = \int d^dx \sqrt{\widehat{\gamma}} L(G;\phi)$. The explicit x-dependence of the sources G allows one to define local composite operators via functional differentiation after renormalization. In addition the scalar source F provides an elegant way to compute the nonlinear renormalizations of the quantum fields in the background expansion [56].

The background field method involves decomposing the fields ϕ^j into a classical background field configuration φ^j and a formal power series in the quantum fields ξ^j whose coefficients are functions of φ^j. The series is defined in terms of the unique geodesic from the point φ in the target manifold to the (nearby) point ϕ, where ξ^j is the tangent vector at φ. We shall write $\phi^j(\varphi;\xi)$ for this series, and refer to ϕ, φ, and ξ as the full field, the background field, and the quantum field, respectively. On the bare level one starts with $\phi^j_B := \phi^j(\varphi_B;\xi_B)$ which upon renormalization are replaced by $\phi^j = \phi^j(\varphi;\xi)$. The transition function $\xi_B(\xi)$ can be computed from the differential operator $Z-1$ in eq. (B.2) below. For our purposes we in addition have to allow for a renormalization $\varphi_B(\varphi)$ of the background fields. As usual we adopt the convention that the fields ϕ^j_B remain dimensionless for base space dimension $d \neq 2$. Then the bare couplings/sources $G^B(\phi_B;x)$ have dimension $[\mu]^{d-2}$ and are expressed as a dimensionless sum of the renormalized $G(\phi;x)$ and covariant counter tensors built from $G(\phi;x)$. A suitable parameterization is

$$\begin{aligned}
g^B_{ij} &= \mu^{d-2}\left[g_{ij} + T_{ij}(g)\right], \\
V^B_{\mu i} &= \mu^{d-2}\left[Z^V(g)_i^j V_{\mu j} + N_i^{\ jk}(g)\overline{\partial}_\mu g_{jk}\right], \\
\Phi^B &= \mu^{d-2}\left[Z(g)\Phi + \Psi(g)\right], \\
F^B &= \mu^{d-2}\left[Z(g)F + Y\right].
\end{aligned} \tag{B.2}$$

Here $\overline{\partial}_\mu$ denotes differentiation with respect to x at fixed ϕ. The quantities T_{ij}, $N_i^{\ jk}$, Ψ, Y and Z^V-1, $Z-1$ contain poles and only poles in $(2-d)$ whose coefficients are defined by minimal subtraction. Except for Y they depend on g_{ij} only; Y in addition depends quadratically on $V_{\mu i}$ and $\overline{\partial}_\mu g_{jk}$, but the quadratic forms with which they are contracted again only depend on g_{ij}. All purely g-dependent counter tensors are algebraic functions of g_{ij}, its covariant derivatives and its curvature tensors. $Z-1$ and Z^V-1 specifically are linear differential operators acting on scalars and vectors on the target manifold, respectively. The combined pole and loop expansion takes the form

$$\mathcal{O} = \sum_{\nu\geq 1}\sum_{l\geq\nu} \frac{1}{(2-d)^\nu}\left(\frac{1}{2\pi}\right)^l \mathcal{O}^{(\nu,l)}, \tag{B.3}$$

for any of the quantities $T_{ij}, N_i^{\ jk}$, Ψ, Z^V-1, $Z-1$, Y. The residue of the simple pole is denoted by $\mathcal{O}^{(1)}$. We do not include explicitly powers of the loop counting parameter λ in (B.3). For the purely g-dependent counter terms of interest here they are easily restored by inserting g/λ and utilizing the scaling properties listed below. However once g is 'deformed' into a nontrivial function of λ the 'scaling decomposition' (B.3) no longer coincides with the expansion in powers of λ and the former is the fundamental one. Under a constant rescaling of the metric the purely g-dependent counter term coefficients transform homogeneously as follows

$$\begin{aligned} \mathcal{O}^{(\nu,l)}(\Lambda^{-1}g) &= \Lambda^{l-1}\mathcal{O}^{(\nu,l)}(g) \qquad \text{for } \mathcal{O} = T_{ij},\, \Psi\,, \\ \mathcal{O}^{(\nu,l)}(\Lambda^{-1}g) &= \Lambda^{l}\, \mathcal{O}^{(\nu,l)}(g) \qquad \text{for } \mathcal{O} = Z,\, Z^V,\, N\,. \end{aligned} \tag{B.4}$$

In principle the higher order pole terms $\mathcal{O}^{(\nu,l)}$, $l \geq \nu \geq 2$, are determined recursively by the residues $\mathcal{O}^{(1,l)}$ of the first order poles via "generalized pole equations". The latter can be worked out in analogy to the quantum field theoretical case; see [53, 57, 59]. Taking the consistency of the cancellations for granted one can focus on the residues of the first order poles, which we shall do throughout.

Explicit results for them are typically available up to and including two loops [52, 58, 57, 56, 59]. For the metric and Φ also the three-loop results are known:[6]

$$\begin{aligned} T_{ij}^{(1,1)}(g) &= R_{ij}\,, \\ T_{ij}^{(1,2)}(g) &= \frac{1}{4} R_{iklm}\, R_j{}^{klm}\,. \\ T_{ij}^{(1,3)}(g) &= \frac{1}{6} R_{imn}{}^{k} R_{jpqk} R^{pnmq} - \frac{1}{8} R_{iklj} R^{k}{}_{mnp} R^{lmnp} - \frac{1}{12} \nabla_n R_{iklm} \nabla^k R_j{}^{lmn}\,, \end{aligned} \tag{B.5}$$

where the three-loop term has been computed independently by Fokas-Mohammedi [61] and Graham [63]. For Φ the results are [54, 59, 57, 58]

$$\Psi^{(1,1)}(g) = \frac{c_T}{6}\,, \qquad \Psi^{(1,2)}(g) = 0\,, \qquad \Psi^{(1,3)}(g) = \frac{1}{48} R_{ijkl} R^{ijkl}\,, \tag{B.6}$$

where c_T is the dimension of the target manifold. For the other quantities one has [59, 56, 58, 62]:

$$\begin{aligned} [Z^V(g)_i^j]^{(1,1)} &= \frac{1}{2}[-\nabla^2 \delta_i^j + R_i^{\,j}]\,, \\ [Z^V(g)_i^j]^{(1,2)} &= \frac{1}{4} R_i{}^{klj} \nabla_k \nabla_l\,, \end{aligned} \tag{B.7a}$$

$$\begin{aligned} Z(g)^{(1,1)} &= -\frac{1}{2}\nabla^2\,, \quad Z(g)^{(1,2)} = 0\,, \\ Z(g)^{(1,3)} &= -\frac{3}{16} R^{iklm} R^{j}{}_{klm} \nabla_i \nabla_j\,. \end{aligned} \tag{B.7b}$$

$$\begin{aligned} [N_i{}^{jk}(g)]^{(1,1)} &= \frac{1}{2}\delta_i^j \nabla^k - \frac{1}{4} g^{jk} \nabla_i\,, \\ [N_i{}^{jk}(g)]^{(1,2)} &= \frac{1}{2} R_i{}^{jkl} \nabla_l\,. \end{aligned} \tag{B.7c}$$

The expressions for $Y^{(1,1)}$ and $Y^{(1,2)}$ are likewise known [59] but are not needed here.

[6] Our conventions are: $\nabla_i v^k = \partial_i v^k + \Gamma^k{}_{ij}\, v^j$, with $\Gamma^k{}_{ij} = \frac{1}{2} g^{kl}[\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}]$. The Riemann tensor is defined by $(\nabla_i \nabla_j - \nabla_j \nabla_i) v^k = R^k{}_{lij}\, v^l$, so that $R^k{}_{lij} = \partial_i \Gamma^k{}_{lj} - \partial_j \Gamma^k{}_{li} + \Gamma^k{}_{im} \Gamma^m{}_{lj} - \Gamma^k{}_{jm} \Gamma^m{}_{li}$. The Ricci tensor is $R_{ij} = R^m{}_{imj}$. For the computation of curvature tensors the maple tensor package is useful. Compared with our conventions one has $R_{ijkl} = (R_{ijkl})^{\text{maple}}$, $R_{ij} = -(R_{ij})^{\text{maple}}$.

JHEP12(2002)066

Some explanatory comments should be added. First, in addition to the minimal subtraction scheme the above form of the counter tensors refers to the background field expansion in terms of riemannian normal coordinates. If a different covariant expansion is used the counter tensors change (see e.g. [54] for a one-loop illustration). Likewise the standard form of the higher pole equations [53, 57, 59] is only valid in a preferred scheme. E.g. for the metric counter terms in this scheme additive contributions to $T_{ij}(g)$ of the form $\mathcal{L}_V g_{ij}$ are absent [56]. Note that adding such a term for $\nu = 1$ leaves the metric beta function in eq. (B.17) below unaffected, provided V^j is functionally independent of g_{ij}.

So far only the full fields entered, $\phi^j_{\scriptscriptstyle B}$ on the bare and ϕ^j on the renormalized level. Their split into background and quantum contributions is however likewise subject to renormalization. A convenient way to determine the transition function $\xi^j_{\scriptscriptstyle B}(\xi)$ from the bare to the renormalized quantum fields was found by Howe, Papadopolous and Stelle [56]. In effect one considers the inversion $\xi^j(\varphi; \phi - \varphi)$ of the normal coordinate expansion $\phi^j = \phi^j(\varphi; \xi)$ of the renormalized fields. If Z in (B.2) is regarded as a differential operator acting on the second argument of this function, i.e. on ϕ,

$$\xi^j(\xi_{\scriptscriptstyle B}) = Z\, \xi^j(\varphi; \phi - \varphi)\,, \tag{B.8}$$

one obtains the desired $\xi^j_{\scriptscriptstyle B}(\xi)$ relation by inversion. To lowest order $Z^{(1,1)} = -\frac{1}{2}\nabla^2$ yields

$$\xi^i_{\scriptscriptstyle B} = \xi^i + \frac{1}{2-d}\frac{\lambda}{2\pi}\left[\frac{1}{3}R^i{}_j\xi^j + \frac{1}{4}\nabla_k R^i{}_j\xi^k\xi^j - \frac{1}{24}\nabla^i R_{kj}\xi^k\xi^j + O(\xi^3)\right]. \tag{B.9}$$

At each loop order the coefficient is a power series in ξ whose coefficients are covariant expressions built from the metric $g_{ij}(\varphi)$ at the background point.

With all these renormalizations performed the result can be summarized in the proposition [56, 59] that the source-extended background functional

$$\exp\Gamma[G;\varphi] = \int [\mathcal{D}\xi]\exp\left\{-S[G_{\scriptscriptstyle B},\phi_{\scriptscriptstyle B}] + \frac{1}{\lambda}\int d^dx J_i(\varphi)\xi^i\right\} \tag{B.10}$$

defines a finite perturbative measure to all orders of the loop expansion. The additional source $J_i(\varphi)$ here is constrained by the requirement that $\langle\xi^j\rangle = 0$. The key properties of $\Gamma(G;\varphi)$ are:

- It is invariant under reparameterizations of the background fields φ.
- It obeys a simple renormalization group equation (which would not be true without the F-source).
- A generalized action principle holds that allows one to construct local composite operators of dimension $0, 1, 2$, by variation with respect to the renormalized sources.

Let $V(\phi)$, $V_i(\phi)$, $V_{ij}(\phi)$ be a scalar, a vector, and a symmetric tensor on the target manifold, respectively. 'Pull-back' composite operators of dimension 0,1,2 are defined by [59]

$$[\![V(\phi)]\!] = \lambda V \cdot \frac{\partial}{\partial F}L_{\scriptscriptstyle B} = \mu^{d-2}Z(g)V\,,$$

$$\llbracket V_i(\phi)\partial^\mu\phi^i\rrbracket = \lambda V_i\cdot\frac{\partial}{\partial V_{\mu i}}L_B = \mu^{d-2}\left[\partial^\mu\phi^i Z^V(g)_i^j V_j + V_i\cdot\frac{\partial}{\partial V_{\mu i}}Y\right], \tag{B.11}$$

$$\left\llbracket\frac{1}{2}V_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j\right\rrbracket = \lambda V\cdot\frac{\partial}{\partial g}L_B - \frac{\mu^{d-2}}{\sqrt{\widehat{\gamma}}}\partial_\mu\left[\sqrt{\widehat{\gamma}}\partial^\mu\phi^i N_i^{jk}(g)\,V_{jk} + \sqrt{\widehat{\gamma}}V_{ij}\cdot\frac{\partial}{\partial(\overline{\partial}_\mu g_{ij})}Y\right].$$

The functional derivatives here act on functionals on the target manifold at fixed x, e.g. $V\cdot\frac{\partial}{\partial F} = \int d^D\phi\sqrt{g}V(\phi;x)\frac{\partial}{\partial F(\phi;x)}$. For g_{ij} in addition the dependence of the counter terms on $\overline{\partial}_\mu g_{ij}$ has to be taken into account, so that $V\cdot\frac{\partial}{\partial g} := V_{ij}\cdot\frac{\partial}{\partial g_{ij}} + \overline{\partial}_\mu V_{ij}\cdot\frac{\partial}{\partial(\overline{\partial}_\mu g_{ij})}$. Further $L_B = L(G_B,\phi_B)$ is the bare lagrangian regarded as a function of the renormalized quantities. The contractions on the base space are with respect to the backgound metric $\widehat{\gamma}_{\mu\nu}$. The additional total divergence in (B.12c) reflects the effect of operator mixing. The normal products as given in (B.12) still refer to the functional measure as defined by the source-extended lagrangian. After all differentiations have been performed the sources should set to zero or rendered x-independent again to get the composite operators e.g. for the purely metric sigma-model.

The definition (B.12) of the normal products is consistent with redefinitions of the couplings/sources that change the lagrangian only by a total divergence. The operative identities are

$$(Z^V)_i^j\partial_j V = \partial_i(ZV)\,, \qquad (\overline{\partial}^\mu Z)V = \partial_i V\cdot\frac{\partial Y}{\partial V_{\mu i}}\,, \tag{B.12}$$

for a scalar $V(\phi;x)$. They entail

$$\partial_\mu\llbracket V\rrbracket = \llbracket\partial_i V\partial_\mu\phi^i\rrbracket + \llbracket\overline{\partial}_\mu V\rrbracket\,. \tag{B.13}$$

Moreover the invariance of the regularization under reparameterizations of the target manifold allows one to convert the reparameterization invariance of the basic lagrangian (B.1) into a “diffeomorphism Ward identity” [55, 59]:

$$\frac{1}{\sqrt{\widehat{\gamma}}}\partial^\mu\llbracket\sqrt{\widehat{\gamma}}\lambda J_\mu(v)\rrbracket = \left\llbracket\frac{1}{2}\mathcal{L}_v g_{ij}\partial^\mu\phi^i\partial_\mu\phi^j + \partial^\mu\phi^i\mathcal{L}_v V_{\mu i} + \frac{1}{2}R^{(2)}(\widehat{\gamma})\mathcal{L}_v\Phi + \mathcal{L}_v F\right\rrbracket - \lambda v^i\cdot\frac{\delta S_B}{\delta\phi^i}\,, \tag{B.14}$$

with

$$\lambda J_\mu(v) = \partial_\mu\phi^i v_i + v^i V_{\mu i}\,.$$

The Lie derivative terms on the right hand side are the response of the couplings/sources under an infinitesimal diffeomorphism $\phi^j\to\phi^j + v^j(\phi)$. Thus $J_\mu(v)$ may be viewed as a “diffeomorphism current”. The last term on the right hand side is the (by itself finite) “equations of motion operator”. In deriving (B.14) the following useful consistency conditions arise

$$\begin{aligned} g_{ij}^B v^j &= \mu^{d-2}\left[Z^V(g)_i^j v_j + N_i^{\,jk}(g)\mathcal{L}_v g_{jk}\right], \\ v^i V_{\mu i}^B &= \mu^{d-2}\left[\mathcal{L}_v g_{ij}\cdot\frac{\partial Y}{\partial(\overline{\partial}^\mu g_{ij})} + v_i\cdot\frac{\partial Y}{\partial V_i^\mu} + Z(v^i V_{\mu i})\right]. \end{aligned} \tag{B.15}$$

So far the renormalization was done at a fixed normalization scale μ. The scale dependence of the renormalized couplings/sources $G = \{g_{ij}, V_{\mu i}, \Phi, F\}$ is governed by a set of renormalization functions which follow from (B.2). For a counter tensor of the form (B.3) it is convenient to introduce

$$\dot{\mathcal{O}} = -\sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l l\,\mathcal{O}^{(1,l)}\,, \tag{B.16}$$

which in view of (B.4) can be regarded as a parametric derivative of $\mathcal{O}^{(1)}$. Then

$$\begin{aligned}
\mu\frac{d}{d\mu}g_{ij} &= \beta_{ij} := (2-d)g_{ij} - \dot{T}_{ij}\,,\\
\mu\frac{d}{d\mu}V_{\mu i} &= \gamma^V := (2-d)V_{\mu i} - (\dot{Z}^V)_i^j V_{\mu j} - \dot{N}_i^{\,jk}\overline{\partial}_\mu g_{jk}\,,\\
\mu\frac{d}{d\mu}\Phi &= \gamma^\Phi := (2-d-\dot{Z})\Phi - \dot{\Psi}\,,\\
\mu\frac{d}{d\mu}F &= \gamma^F := (2-d-\dot{Z})F - \dot{Y}\,.
\end{aligned} \tag{B.17}$$

The associated renormalization group operator is

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta\cdot\frac{\partial}{\partial g} + \gamma_\mu^V\cdot\frac{\partial}{\partial V_\mu} + \gamma^\Phi\cdot\frac{\partial}{\partial\Phi} + \gamma^F\cdot\frac{\partial}{\partial F}\,. \tag{B.18}$$

For example the dimension 0 composite operators in (B.12) obey

$$\mathcal{D}[\![V(\phi)]\!] = [\![(d-2+\dot{Z}+\mathcal{D})V]\!]\,, \tag{B.19}$$

and similar equations hold for the dimension 1,2 composite operators.

An important application of this framework is the determination of the Weyl anomaly as an ultraviolet finite composite operator. We shall only need the version without vector and scalar functionals. The result then reads [55, 58, 59]

$$\widehat{\gamma}^{\mu\nu}[\![T_{\mu\nu}]\!] = \frac{1}{2}\left[\!\!\left[B_{ij}\left(\frac{g}{\lambda}\right)\widehat{\gamma}^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right]\!\!\right] + \frac{1}{2}R^{(2)}(\widehat{\gamma})[\![B^\Phi]\!]\,. \tag{B.20}$$

Here the so-called Weyl anomaly coefficients enter:

$$\begin{aligned}
\lambda B_{ij}\left(\frac{g}{\lambda}\right) &:= \lambda\beta_{ij}\left(\frac{g}{\lambda}\right)\Big|_{d=2} + \mathcal{L}_S g_{ij}\,,\\
\lambda B^\Phi\left(\Phi,\frac{g}{\lambda}\right) &:= \lambda\gamma^\Phi\left(\frac{g}{\lambda}\right)\Big|_{d=2} + S^j\partial_j\Phi\,,
\end{aligned} \tag{B.21}$$

where β_{ij} and γ^Φ are the renormalization group functions of eq. (B.17) and

$$S_i := W_i + \partial_i\Phi \tag{B.22}$$

with

$$W_i := N^{(1)}(g)_i^{\,jk}g_{jk} = \left(\frac{\lambda}{2\pi}\right)^3\frac{1}{32}\partial_i(R_{klmn}R^{klmn}) + O(\lambda^4)\,.$$

These expressions hold in dimensional regularization, minimal subtraction, and the backgound field expansion in terms of normal coordinates. Terms proportional to the equations of motion operator $\delta S_B/\delta\phi^j$ have been omitted. The normal-products (B.12) are normalized such that the expectation value of an operator contains as its leading term the value of the corresponding functional on the background, $\langle\mathcal{O}(\phi)\rangle = \mathcal{O}(\varphi) + \cdots$, where the subleading terms are in general nonlocal and depend on the scale μ. For the expectation value of the trace anomaly this produces a rather cumbersome expression, see e.g. [58]. As stressed in [55] the result (B.20), in contrast, allows one to use $B_{ij}(g) = 0$ as a simple criterion to select functionals which 'minimize' the conformal anomaly.

The Weyl anomaly coefficients (and the anomaly itself) can be shown to be invariant under field redefinitions of the form

$$\phi_B^j \longrightarrow \phi_B^j + \frac{1}{2-d} V^j(\phi,\lambda)\,, \tag{B.23}$$

with $V^j(\phi,\lambda) = \sum_{l\geq 1}(\frac{\lambda}{2\pi})^l V_l^j(\phi)$ functionally *independent* of the metric. Roughly speaking (B.23) changes the beta function by a Lie derivative term that is compensated by a contribution of the diffeomorphism current to the anomaly which amounts to $W_j \to W_j - V_j$ [55]. It is important to distinguish these diffeomorphisms from field renormalizations like (5.25) that depend on the metric. Although formally (B.23) amounts to $\Xi^j(\phi,\lambda) \longrightarrow \Xi^j(\phi,\lambda) + V^j(\phi,\lambda)$ in (5.25), clearly one cannot cancel one against the other. The distinction is also highlighted by considering the change in the metric counter terms

$$T_{ij}^{(1)}(g) \longrightarrow T_{ij}^{(1)}(g) - \mathcal{L}_V g_{ij}\,, \tag{B.24}$$

under (B.23). Without further specifications this would not be legitimate for a g-dependent vector. Although the Lie derivative term in (B.24) drops out when recomputing β_{ij} directly as a parametric derivative, in combinations like

$$\beta_{ij}(\phi_B)\partial^\mu\phi_B^i\partial_\mu\phi_B^j = \beta_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j + \frac{1}{2-d}\mathcal{L}_V\beta_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j + \cdots\,, \tag{B.25}$$

the term $(2-d)g_{ij}$ in the metric beta function of (B.17) induces an effective shift

$$\beta_{ij}(g) \longrightarrow \beta_{ij}(g) + \mathcal{L}_V g_{ij}\,. \tag{B.26}$$

Similarly W_i is shifted to $W_i - V_i$ and the Weyl anomaly coefficients are invariant.

In the context of riemannian sigma-models Φ is usually interpreted as a "string dilaton" for the systems (B.1) defined on a curved base space. If one is interested in the renormalization of (B.1) on a flat base space, Φ on the other hand plays the role of a potential for the improvement term of the energy momentum tensor. This role of Φ can be made manifest by rewriting (B.20) by means of the diffeomorphism Ward identity. Returning to a flat base space one finds [55, 59]

$$[\![T^\mu_{\ \mu}]\!] = \frac{1}{2}\left[\!\!\left[\beta_{ij}\left(\frac{g}{\lambda}\right)\partial^\mu\phi^i\partial_\mu\phi^j\right]\!\!\right] + \partial^\mu\partial_\mu[\![\Phi]\!] + \partial^\mu[\![\partial_\mu\phi^i W_i]\!]\,, \tag{B.27}$$

where again terms proportional to the equations of motion operator have been omitted. Here $\partial^2[\![\Phi]\!]$ is the 'naive' improvement term while the additional total divergence is induced by operator mixing.

The functions B^Φ and $B_{ij}(g)$ are linked by an important consistency condition, the Curci-Paffuti relation [57]. We present it in two alternative versions

$$\partial_i \dot\Psi = \dot N_i^{\;jk}\, \dot T_{jk} - \dot T \cdot \frac{\partial}{\partial g} W_i + (\dot Z^V)_i^j W_j\,, \tag{B.28a}$$

$$\partial_i B^\Phi = \dot N_i^{\;jk}\, B_{jk} - B \cdot \frac{\partial}{\partial g} S_i + B_{ij} S^j\,. \tag{B.28b}$$

The first version displays the fact that the identity relates various g-dependent counter terms without Φ entering. In the second version Φ is introduced in a way that yields an identity among the Weyl anomaly coefficients. It has the well-known consequence that B^Φ is constant when B_{ij} vanishes:

$$B_{ij} = 0 \quad \Longrightarrow \quad B^\Phi = \frac{c_T}{6}\,, \tag{B.29}$$

where c_T is the central charge of energy momentum tensor derived from (B.1).

C. Strict renormalizability yields no fixed point

Although the fixed point $h^{\rm beta}$ was constructed in a loopwise expansion it can be regarded as "non-gaussian" in the following sense: There exists a function $h^{\rm ren}(\,\cdot\,,\lambda)$ for which $h_{\scriptscriptstyle B}^{\rm ren}(\,\cdot\,,\lambda) = h^{\rm ren}(\,\cdot\,,\lambda)$, so that one almost recovers conventional renormalizability.[7] However beyond one loop $h^{\rm ren}$ *differs* from $h^{\rm beta}$, in particular $\beta_h(h^{\rm ren}) \neq 0$. Thus if one was to explore the vicinity of this conventionally renormalizable theory the fixed point $h^{\rm beta}$ could not be seen and in view of eq. (6.8) their was little chance to impose the operator constraints.

In order to find $h^{\rm ren}$ we return to the general solution of the finiteness condition (4.13) with $h(\rho,\lambda)$ of the form (4.10). On can then ask whether there exist special choices for the functions $h_l(\rho)$, $l \geq 0$, such that $H(\rho,\lambda)$ is ρ-independent. This requirement translates into a system of second order differential equations that can be recursively solved for h_0, h_1, etc.

$$\begin{aligned} &H(\rho,\lambda) = -Z(\lambda) = -\sum_{l\geq 1} z_l \left(\frac{\lambda}{2\pi}\right)^l \quad \Longleftrightarrow \\ &\frac{h(\rho\partial_\rho)^2 h}{(\rho\partial_\rho h)^2} = 1 + \frac{B_\lambda(\lambda/h) - \beta_\lambda(\lambda/h)h/\lambda}{B_\lambda(\lambda/h) + Z(\lambda)}\,. \end{aligned} \tag{C.1}$$

In this situation one almost recovers conventional renormalizability: The bare and the renormalized metric in (4.12) are related by a *numerical* though singular prefactor, which can be attributed in the usual way to a renormalization of the coupling

$$\lambda_{\scriptscriptstyle B} = \mu^{2-d}\lambda\left[1 + \frac{1}{2-d} Z(\lambda) + \cdots\right]. \tag{C.2}$$

[7] I thank P. Forgács for pointing this out. A similar concept of recovering renormalizability by finite quantum deformations was recently employed in the context of T-duality [48]; see also [49].

Combined with the (still in general nonlinear) field redefinitions all counter terms can be absorbed. The price to pay is that the 'good' target space geometry in which the bare and the renormalized lagrangian have the same functional form is not known a-priori. The functions h_1, h_2, etc, constitute *finite quantum deformations* of the naive target space metric. Judiciously chosen they ensure conventional renormalizability, but the proper choice has to be determined order by order by solving (C.1). At the one-loop level one finds that the only solutions of $H_1[h_0] = -z_1$ are again powers of ρ, with the power independent of z_1. A non-zero z_1 can arise either for the constant solution, $h_0 = -\zeta_1/z_1$, or when a finite integration constant in (4.6) is permitted. Then $H_1[\rho^p] = \zeta_1 \rho_1^{-p}$, and with the choice $\rho_1 = \infty$ advocated before only the solution ρ^p, $p > 0$, (with $z_1 = 0$) are left. For $l \geq 1$ the solutions for h_l will have a nontrivial dependence on $z_1, \ldots, z_l$. However they can not be chosen so as to make h_l vanish, i.e. the required quantum deformations are always non-trivial. Since on the other hand the structure of the solutions considerably simplifies if the z_l's are chosen to vanish, we consider only this case in the following. The solutions for h_l will then still depend on l arbitrary deformation parameters; switching on the z_l's simply enlarges the number of parameters at the expense of the cumbersome coupling renormalization (C.2).

The case $H(\rho, \lambda) = 0$ has the bonus that the field renormalization vectors $\Xi^j(\rho, \lambda)$ are gradients of a potential:

$$
\begin{aligned}
&H(\rho,\lambda) = 0 \quad \Longleftrightarrow \quad h(\rho,\lambda)\Xi^3(\rho,\lambda) = \rho\frac{\lambda}{2\pi}C(\lambda)^{-1} \quad \Longleftrightarrow \\
&\Xi^j = -\partial^j \Phi\,, \quad \text{with} \quad \Phi = \sum_{l\geq 1}\left(\frac{\lambda}{2\pi}\right)^l \Phi_l\,,
\end{aligned} \tag{C.3}
$$

for some $C(\lambda) = \sum_{l\geq 0} C_l (\lambda/2\pi)^l$ with constant C_l. The potential is given by

$$
\Phi^{\rm ren}(\rho,\sigma,\lambda) = -\frac{\lambda}{4\pi}C(\lambda)^{-1}(a\ln\rho + 2b\sigma) - \frac{1}{2}\int^{\rho}\frac{du}{u}h(u,\lambda)\int^{u}\frac{dv}{v}S(v,\lambda)\,. \tag{C.4}
$$

The ambiguity stemming from the various integration constants is again of the form (6.7).

The condition (C.3) converts into the first order differential equation

$$
\frac{\lambda}{2\pi}\rho\partial_\rho h(\rho,\lambda) = C(\lambda)h(\rho,\lambda)^2 B_\lambda\left(\frac{\lambda}{h(\rho,\lambda)}\right), \tag{C.5}
$$

from which upon insertion of (4.10) the functions h_l, $l \geq 0$, are determined recursively. We denote the solutions by $h_l^{\rm ren}$. There exists a minimal solution corresponding to a λ-independent $C(\lambda) = C_0$. It comes out as

$$
h^{\rm ren}(\rho,\lambda) = \rho^p - \frac{\lambda}{2\pi}\frac{\zeta_2}{\zeta_1} - \left(\frac{\lambda}{2\pi}\right)^2\frac{\zeta_3}{2\zeta_1}\rho^{-p} + \cdots\,, \tag{C.6}
$$

where $C_0 = p/\zeta_1$. Switching on the parameters $C_1, C_2, \ldots$, leads to a deformation of the functions $h_1, h_2, \ldots$, similar to the one in (5.18). Indeed, in view of (4.17), the solutions of (C.5) must be related to those of (5.15) by substituting $\zeta_l \to \zeta_l/l$.

Let us now compare $h^{\rm ren}$ with $h^{\rm beta}$ defined through the vanishing of the $\beta_h(h)$ functional. Already for the minimal solution one sees from (5.16) that

$$h_l^{\rm ren}(\rho) \neq h_l^{\rm beta}(\rho)\,, \qquad l \geq 2\,. \tag{C.7}$$

Note that the disagreement starts at the two-loop level where the coefficients ζ_1, ζ_2 are still universal. In particular

$$\lambda\beta_h(h^{\rm ren}/\lambda) = \left(\frac{\lambda}{2\pi}\right)^2 \frac{\zeta_2}{2\rho^p} + O(\lambda^3)\,, \tag{C.8}$$

cannot be made to vanish by a change of scheme. In section 6 we related the $h_l^{\rm beta}$ class to a physical requirement — the vanishing of the trace anomaly — which is a necessary condition for conformal invariance. The mismatch (C.7) therefore implies that one cannot have both desirable properties, conformal invariance and strict renormalizability, at the same time. Since this is an important conclusion, we made sure that it is not an artifact of some inessential assumption. For example at first sight it seems that by using the more general general target space metric (3.6) (where the residual freedom in choosing adapted coordinates has not been used to simplify the functions $a(\rho)$, $b(\rho)$) the additional freedom to adjust $a(\rho)$ and $b(\rho)$ as functions of λ could be used to compensate a mismatch $h_l^{\rm ren}(\rho) \neq h_l^{\rm beta}(\rho)$. However one can check that this is not the case by repeating the entire construction in this more general setting.

It may be useful to summarize and juxtapose the key relations defining the "renormalizable" h and Φ and that satisfying the "beta" condition. As alternative defining relations one may take

$$\mathcal{L}_{\Xi} g_{ij} = -2\nabla_i\nabla_j\Phi \qquad \text{for "ren"} \tag{C.9a}$$

$$\mathcal{L}_{\dot{\Xi}-W} g_{ij} = 2\nabla_i\nabla_j\Phi \qquad \text{for "beta"} \tag{C.9b}$$

In both cases the condition (C.9) has an unexpected spin-off:

$$H(\rho,\lambda) = 0\,, \quad \text{i.e.} \quad \frac{2\pi}{\lambda}\frac{\Xi^3}{\rho} = \frac{1}{C(\lambda)h}\,, \qquad \text{for "ren"} \tag{C.10a}$$

$$\beta_h(h) = 0\,, \quad \text{i.e.} \quad -\frac{2\pi}{\lambda}\frac{\dot{\Xi}^3}{\rho} = \frac{1}{C(\lambda)h}\,, \qquad \text{for "beta"} \tag{C.10b}$$

Combining (6.1) with (C.9a) and (C.10a) the finiteness condition (4.13) can be rewritten as $2\partial^\mu\partial_\mu\Phi = \lambda T_{ij}^{(1)}(g/\lambda)\partial^\mu\phi^i\partial_\mu\phi^j$, modulo terms proportional to the equations of motion operator. For $h = h^{\rm ren}$ the counter terms for the lagrangian can therefore to all loop orders be written as $\partial^2\Phi$. At 1-loop this also entails the vanishing of the trace anomaly in agreement with the criterion in [54]. At higher loops however this equivalence breaks down.

In summary, the class of h_l's that ensure (almost) conventional renormalizability is *not* the same as the one that ensures the vanishing of the $\beta_h(h)$ function. Likewise the improvement potential $\Phi^{\rm trace} = \Phi^{\rm beta}$ for the energy momentum tensor at the fixed point differs from $\Phi^{\rm ren}$.

References

[1] S. Weinberg, *Ultraviolet divergencies in quantum theories of gravitation*, in *General relativity, an Einstein Centenary survey*, S. Hawking and W. Israel eds., Cambridge University Press, Cambridge 1979.

[2] G. 't Hooft and M. Veltman, One loop divergencies in the theory of gravitation, Ann. Inst. Poincaré **20** (1974) 69.

[3] M. Goroff and A. Sagnotti, *The ultraviolet behavior of Einstein gravity*, *Nucl. Phys.* **B 266** (1986) 709

[4] A. van de Ven, *Two-loop quantum gravity*, *Nucl. Phys.* **B 378** (1992) 309.

[5] J. Ambjorn, *Simplicial euclidean and lorentzian quantum gravity*, `gr-qc/0201028`.

[6] J.C. Baez, J.D. Christensen, T.R. Halford and D.C. Tsang, *Spin foam models of riemannian quantum gravity*, *Class. and Quant. Grav.* **19** (2002) 4627 [`gr-qc/0202017`].

[7] C. Fleischhack and J. Lewandowski, *Breakdown of the action method in gauge theories*, `math-ph/0111001`.

[8] T. Thiemann, *Complexifier coherent states for quantum general relativity*, `gr-qc/0206037`.

[9] M. Reuter, *Nonperturbative evolution equations for quantum gravity*, *Phys. Rev.* **D 57** (1998) 971 [`hep-th/9605030`].

[10] O. Lauscher and M. Reuter, *Ultraviolet fixed point and generalized flow equation of quantum gravity*, *Phys. Rev.* **D 65** (2002) 025013 [`hep-th/0108040`].

[11] O. Lauscher and M. Reuter, *Is quantum Einstein gravity nonperturbatively renormalizable?*, *Class. and Quant. Grav.* **19** (2002) 483 [`hep-th/0110021`].

[12] W. Souma, *Non-trivial ultraviolet fixed point in quantum gravity*, *Prog. Theor. Phys.* **102** (1999) 181 [`hep-th/9907027`].

[13] S. Falkenberg and S.D. Odintsov, *Gauge dependence of the effective average action in Einstein gravity*, *Int. J. Mod. Phys.* **A 13** (1998) 607 [`hep-th/9612019`].

[14] R. Percacci and D. Perini, *Constraints on matter from asymptotic safety*, `hep-th/0207033`.

[15] N. Bjerrum-Bohr, J. Donoghue and B. Holstein, *Quantum gravitational corrections to the nonrelativistic scattering of two masses*, `hep-th/0211072`.

[16] P.O. Mazur and E. Mottola, *The gravitational measure, solution of the conformal factor problem and stability of the ground state of quantum gravity*, *Nucl. Phys.* **B 341** (1990) 187.

[17] P. Forgacs and M. Niedermaier, *A fixed point for truncated quantum Einstein gravity*, `hep-th/0207028`.

[18] R. Matzner and C. Misner, *Gravitational field for sources with axial symmetry and angular momentum*, *Phys. Rev.* **154** (1967) 1229.

[19] F. Ernst, *New formulation of the axially symmetric gravitational field problem*, *Phys. Rev.* **167** (1968) 1175.

[20] D. Kramer and G. Neugebauer, *Zu axialsymmetrischen stationären Lösungen der Einstein'schen Feldgleichungen für das Vakuum*, *Commun. Math. Phys.* **10** (1968) 132.

[21] D. Maison, *Duality and hidden symmetries in gravitational theories*, in *Lecture Notes Phys.* **540** (2000) 273.

[22] M.E. Fels and C.G. Torre, *The principle of symmetric criticality in general relativity*, *Class. and Quant. Grav.* **19** (2002) 641 [gr-qc/0108033].

[23] J.D. Romano and C.G. Torre, *Internal time formalism for spacetimes with two killing vectors*, *Phys. Rev.* **D 53** (1996) 5634 [gr-qc/9509055].

[24] B. Julia and H. Nicolai, *Conformal internal symmetry of 2d sigma-models coupled to gravity and a dilaton*, *Nucl. Phys.* **B 482** (1996) 431 [hep-th/9608082].

[25] J. Wainwright, W. Ince and B. Marshman, *Spatially homogenous and inhomogenous cosmologies with equation of state* $p = \mu$, *Gen. Rel. Grav.* **10** (1979) 259.

[26] V. Faraoni, E. Gunzig and P. Nardone, *Conformal transformations in classical gravitational theories and in cosmology*, *Fund. Cosmic Phys.* **20** (1999) 121 [gr-qc/9811047].

[27] H. Nicolai, D. Korotkin and H. Samtleben, *Integrable classical and quantum gravity*, hep-th/9612065.

[28] M. Niedermaier and H. Samtleben, *An algebraic bootstrap for dimensionally reduced quantum gravity*, *Nucl. Phys.* **B 579** (2000) 437 [hep-th/9912111].

[29] D. Korotkin and H. Samtleben, *Yangian symmetry in integrable quantum gravity*, *Nucl. Phys.* **B 527** (1998) 657 [hep-th/9710210].

[30] K. Kuchař, *Canonical quantization of cylindrical gravitational waves*, *Phys. Rev.* **D 4** (1971) 955.

[31] C.G. Torre, *A complete set of observables for cylindrically symmetric gravitational fields*, *Class. and Quant. Grav.* **8** (1991) 1895.

[32] A. Ashtekar and M. Pierri, *Probing quantum gravity through exactly soluble midi-superspaces, 1*, *J. Math. Phys.* **37** (1996) 6250 [gr-qc/9606085].

[33] A. Ashtekar, *Large quantum gravity effects: unforseen limitations of the classical theory*, *Phys. Rev. Lett.* **77** (1996) 4864 [gr-qc/9610008].

[34] M. Niedermaier, *Quantized Einstein-rosen waves,* AdS_2 *and spontaneous symmetry breaking*, *Phys. Lett.* **B 498** (2001) 83 [hep-th/0007227].

[35] D. Korotkin and H. Nicolai, *Isomonodromic quantization of dimensionally reduced gravity*, *Nucl. Phys.* **B 475** (1996) 397 [hep-th/9605144].

[36] B. de Wit, M.T. Grisaru, H. Nicolai and E. Rabinovici, *Two loop finiteness of* $d = 2$ *supergravity*, *Phys. Lett.* **B 286** (1992) 78 [hep-th/9205012].

[37] Y. Satoh, *Field theoretical quantum effects on the Kerr geometry*, *Phys. Rev.* **D 53** (1996) 7026 [hep-th/9512019].

[38] A.A. Tseytlin, *Finite sigma models and exact string solutions with minkowski signature metric*, *Phys. Rev.* **D 47** (1993) 3421 [hep-th/9211061].

[39] A.A. Tseytlin, *String vacuum backgrounds with covariantly constant null killing vector and 2-d quantum gravity*, *Nucl. Phys.* **B 390** (1993) 153 [hep-th/9209023].

[40] T. Damour and A.M. Polyakov, *The string dilaton and a least coupling principle*, *Nucl. Phys.* **B 423** (1994) 532 [hep-th/9401069].

[41] T. Damour and A.M. Polyakov, *The string dilaton and a least coupling principle*, *Nucl. Phys.* **B 423** (1994) 532 [hep-th/9401069].

[42] E. Brezin and S. Hikami, *Three loop calculations in the two-dimensional nonlinear sigma-model*, *J. Phys.* **A 11** (1978) 1141.

[43] S. Hikami, *Three loop beta-functions of nonlinear sigma models on symmetric spaces*, *Phys. Lett.* **B 98** (1981) 208.

[44] W. Bernreuther and F.J. Wegner, *Four loop order beta function for two-dimensional nonlinear sigma models*, *Phys. Rev. Lett.* **57** (1986) 1383.

[45] F. Wegner, *Four loop order beta function of nonlinear sigma models in symmetric spaces*, *Nucl. Phys.* **B 316** (1989) 663.

[46] C. Becchi, A. Blasi, G. Bonneau, R. Collina and F. Delduc, *Renormalizability and infrared finiteness of nonlinear sigma models: a regularization independent analysis for compact coset spaces*, *Commun. Math. Phys.* **120** (1988) 121.

[47] J. Balog, P. Forgacs, Z. Horvath and L. Palla, *Quantum corrections of abelian duality transformations*, *Phys. Lett.* **B 388** (1996) 121 [hep-th/9606187].

[48] G. Bonneau and P.-Y. Casteill, *Dualised sigma-models at the two-loop order*, *Nucl. Phys.* **B 607** (2001) 293 [hep-th/0103260].

[49] G. Bonneau, *Renormalizability and nonproduction in complex sine-gordon model*, *Phys. Lett.* **B 133** (1983) 341.

[50] M. Lüscher, *Quantum nonlocal charges and absence of particle production in the two-dimensional nonlinear sigma model*, *Nucl. Phys.* **B 135** (1978) 1.

[51] A. Montanari, *Non-perturbative renormalization in lattice field theory*, hep-lat/0104005.

[52] D.H. Friedan, *Nonlinear models in two + ϵ dimensions*, *Ann. Phys. (NY)* **163** (1985) 318.

[53] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model*, *Ann. Phys. (NY)* **134** (1981) 85.

[54] C.M. Hull and P.K. Townsend, *Finiteness and conformal invariance in nonlinear sigma models*, *Nucl. Phys.* **B 274** (1986) 349.

[55] G. Shore, *A local renormalization group equation, diffeomorphisms and conformal invariance in sigma models*, *Nucl. Phys.* **B 286** (1987) 349.

[56] P. Howe, G. Papadopoulos and K. Stelle, *Background field method and the nonlinear σ-model*, *Nucl. Phys.* **B 296** (1988) 26.

[57] G. Curci and G. Paffuti, *Consistency between the string background field equation of motion and the vanishing of the conformal anomaly*, *Nucl. Phys.* **B 286** (1987) 399.

[58] A.A. Tseytlin, *Sigma model Weyl invariance conditions and string equations of motion*, *Nucl. Phys.* **B 294** (1987) 383.

[59] H. Osborn, *Renormalization and composite operators in nonlinear sigma models*, *Nucl. Phys.* **B 294** (1987) 595.

[60] H. Osborn, *Renormalization group and two point functions in bosonic sigma models*, *Phys. Lett.* **B 214** (1988) 555.

JHEP12(2002)066

[61] A.P. Foakes and N. Mohammedi, *Three loop calculation of the beta function for the purely metric nonlinear sigma model*, *Phys. Lett.* **B 198** (1987) 359.

[62] I. Jack, D.R.T. Jones and N. Mohammedi, *A four loop calculation of the metric beta function for the bosonic sigma model and the string effective action*, *Nucl. Phys.* **B 322** (1989) 431.

[63] S.J. Graham, *Three loop beta function for the bosonic nonlinear sigma model*, *Phys. Lett.* **B 197** (1987) 543.

[64] J. Polchinski, *Scale and conformal invariance in quantum field theory*, *Nucl. Phys.* **B 303** (1988) 226.

[65] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59; *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466.

[66] E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, *An introduction to T duality in string theory*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 1 [hep-th/9410237].

[67] D. Cangemi, R. Jackiw and B. Zwiebach, *Physical states in matter coupled dilaton gravity*, *Ann. Phys. (NY)* **245** (1996) 408 [hep-th/9505161].

[68] H. Nicolai, *The integrability of $N = 16$ supergravity*, *Phys. Lett.* **B 194** (1987) 402.

[69] I. Bakas, *$O(2,2)$ transformations and the string Geroch group*, *Nucl. Phys.* **B 428** (1994) 374 [hep-th/9402016].

[70] I. Bakas, *Solitons of axion-dilaton gravity*, *Phys. Rev.* **D 54** (1996) 6424 [hep-th/9605043].

[71] P. Breitenlohner, D. Maison and G.W. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, *Commun. Math. Phys.* **120** (1988) 295.

[72] S. Penati, A. Santambrogio and D. Zanon, *Renormalization group flows in sigma-models coupled to two-dimensional dynamical gravity*, *Nucl. Phys.* **B 483** (1997) 495 [hep-th/9605124].

[73] I.R. Klebanov, I.I. Kogan and A.M. Polyakov, *Gravitational dressing of renormalization group*, *Phys. Rev. Lett.* **71** (1993) 3243 [hep-th/9309106].

JHEP12(2002)066